Renormalization of the effective theory for heavy quarks at small velocity

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Abstract

The slope of the Isgur-Wise function at the normalization point, $\xi^{(1)}(1)$, is one of the basic parameters for the extraction of the CKM matrix element V_{cb} from exclusive semileptonic decay data. A method for measuring this parameter on the lattice is the effective theory for heavy quarks at small velocity v. This theory is a variant of the heavy quark effective theory in which the motion of the quark is treated as a perturbation. In this work we study the lattice renormalization of the slow heavy quark effective theory. We show that the renormalization of $\xi^{(1)}(1)$ is not affected by ultraviolet power divergences, implying no need of difficult non-perturbative subtractions. A lattice computation of $\xi^{(1)}(1)$ with this method is therefore feasible in principle. The one-loop renormalization constants of the effective theory for slow heavy quarks are computed to order v^2 together with the lattice-continuum renormalization constant of $\xi^{(1)}(1)$.

We demonstrate that the expansion in the heavy-quark velocity reproduces correctly the infrared structure of the original (non-expanded) theory to every order. We compute also the one-loop renormalization constants of the slow heavy quark effective theory to higher orders in v^2 and the lattice-continuum renormalization constants of the higher derivatives of the ξ function. Unfortunately, the renormalization constants of the higher derivatives are affected by ultraviolet power divergences, implying the necessity of numerical non-perturbative subtractions. The lattice computation of higher derivatives of the Isgur-Wise function seems therefore problematic.

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1 Introduction

The effective theory for heavy quarks ($HQET$) [1, 2] (for a comprehensive review and references to the original literature see ref.[3]) allows a clean determination of the Cabibbo-Kobayashi-Maskawa matrix element $\mid V_{cb} \mid$ from the exclusive semileptonic decays of B mesons

$$B \;\rightarrow\; D^{(*)} + l + \nu_l \tag{1}$$

recently measured by the ARGUS [4] and CLEO [5] collaborations.

In the limit of infinite mass for the charm and the beauty quark,

$$m_c,\; m_b \rightarrow \infty, \tag{2}$$

the six form factors parametrizing the hadronic matrix elements of the decays (1) can all be expressed in terms of a unique form factor, the Isgur-Wise function $\xi(v \cdot v')$ [6, 7, 8],

$$\langle D, v \mid J_\mu^{b\rightarrow c}(0) \mid B, v' \rangle \;=\; \sqrt{M_D M_B}\;(v_\mu + v'_\mu)\;\xi(v \cdot v') \tag{3}$$

$$\langle D^*, v, \epsilon \mid J_\mu^{b\rightarrow c}(0) \mid B, v' \rangle \;=\; -\sqrt{M_D M_B}\;[i\epsilon_{\mu\nu\alpha\beta}\epsilon^\nu {v'}^\alpha v^\beta + \epsilon_\mu(1 + v \cdot v') - v_\mu v' \cdot \epsilon]\xi(v \cdot v') \tag{4}$$

where v' and v denote respectively the b and c quark 4-velocities and $J_\mu^{b\rightarrow c}(x)$ is the weak current describing the transition of a beauty quark into a charm quark, $J_\mu^{b\rightarrow c}(x) = \overline{c}(x)\gamma_\mu(1-\gamma_5)b(x)$. This function is normalized at zero recoil

$$\xi(v' \cdot v = 1) \;=\; 1, \tag{5}$$

and $1/m$-corrections vanish in this kinematical point [9]. A model independent analysis of the decays (1) extrapolates the experimental data up to the endpoint, where the form factors are known by symmetry. In order to eliminate the systematic errors introduced by the extrapolation, it is essential to know also the derivative of the Isgur-Wise function at the normalization point, $\xi^{(1)}(1)$.

So far, four methods have been proposed to compute on the lattice the slope of the Isgur-Wise function. The first one was the estimation of the derivatives of the Isgur-Wise function at the zero-recoil point from the lattice determination of the Isgur-Wise function at discrete points in a region close to the zero-recoil point [10]. This method presents some problems related to the extrapolation to the zero-recoil point that could lead to uncertainties in the determination of $\xi^{(1)}(1)$. The authors of [11] have suggested a new method to compute directly on the lattice the slope of the Isgur-Wise function which avoids any kind of extrapolation. They proposed to study the first spatial moments of two- and three-point meson correlators. Then the derivatives of the Isgur-Wise function could be extracted by forming appropriate ratios of these correlators. The UKQCD Collaboration has recently carried out a exploratory study of the feasibility of this method [12]. The main conclusion is that there are large finite-volume effects in the lattice evaluation of the moments of the correlation functions, having a geometrical origin. Therefore, by increasing the length of the lattice in the spatial directions, these undesiderable volume effects can be reduced. Unfortunately, they are large on currently available lattices. Some approximations have been presented by this group in order to control the volume effects and extract the slope on finite volumes.

Both computations described above treat the heavy quark as an ordinary quark but with a small hopping constant. The first calculation of the Isgur-Wise function using the lattice $HQET$ has been done by the authors of [13]. In that work, the lattice propagator of the heavy quark with velocity v is obtained from a Wick rotated lagrangian [14].

The fourth method is based on an expansion of the $HQET$ in the heavy-quark velocity around the static theory (hereafter called 'effective theory for slow heavy quarks', $SHQET$) [17]. As in the method of spatial moments [11], the derivatives of the Isgur-Wise function at the zero-recoil point can be extracted directly from ratios of two- and three-point correlation functions (see Section 2 for details). The main point is that there are no unexpected geometrical volume effects in the lattice computation of these correlators because the static propagator is local in space. Moreover, the $SHQET$ circumvents the problem of the euclidean continuation of the Georgi theory for heavy-quarks with non-vanishing velocity [14, 15]. In fact, in order to simulate heavy-quark with velocity v on the lattice, the continuum Minkowskian $HQET$ must be transformed into a discretized euclidean field theory. The analytical continuation is not simple because the energy spectrum is unbounded from below [14, 15, 16]. On the contrary, expanding around small velocities, we are perturbing the static theory whose energy spectrum is bounded from below. Roughly speaking, we may say that the heavy quark has a 'perturbative motion' in the $SHQET$ produced by the 'velocity operator' $(\vec{v} \cdot \vec{D})$. This theory has not been used yet in numerical lattice simulations.

In this paper, we analyse the lattice version of the $SHQET$. One of our main results is that the lattice renormalization constant of $\xi^{(1)}(1)$ does not contain any ultraviolet power divergence (i.e. proportional to $1/a^n$, where a is the lattice spacing). The renormalization of $\xi^{(1)}(1)$ is affected only by logarithmic divergences (of the form $\log am$) which can be subtracted with ordinary perturbative computations. This implies that the lattice computation of $\xi^{(1)}(1)$ by simulations using the $SHQET$ is feasible in principle. Ultraviolet power divergences are indeed a serious problem for numerical simulations, because they cannot be subtracted perturbatively with adequate accuracy [18, 19]. We also compute the one-loop lattice renormalization constant of $\xi^{(1)}(1)$. The knowledge of this renormalization constant is essential for converting the values of $\xi^{(1)}(1)$ computed on the lattice to the values in the original (high-energy) theory. Moreover it is shown that the infrared as well as the ultraviolet behaviour of the non-expanded theory ($HQET$) are reproduced by the $SHQET$ order by order in the velocity. This is a non trivial check of the consistency of our approach. We also give the lattice renormalization constants of higher derivatives of the Isgur-Wise function, $\xi^{(n)}(1)$, $n > 1$. Unfortunately, the lattice renormalization constants of higher derivatives $\xi^{(n)}(1)$ for $n > 1$ are affected by power divergences. The computation of higher derivatives with $SHQET$ is therefore more difficult than that one of $\xi^{(1)}(1)$.

This paper is organized as follows. In section 2 we review the $SHQET$ in the continuum [17]; the derivatives of the Isgur-Wise function are expressed as ratios of three- and two-point correlation functions. Section 3 deals with the lattice regularization of the $SHQET$. In section 4 we briefly review the matching theory of lattice operators onto the continuum ones. In section 5 we renormalize the lattice $SHQET$ at order α_s and to all orders in the velocity. In section 6 the renormalization of the heavy quark current $J_\mu^{b\to c}$ is computed. In section 7 we calculate the lattice-continuum renormalization constants for the derivatives of the Isgur-Wise function. Section 8 deals with the problem of power divergences. Finally, in section 9 we present our conclusions. There are also two appendices where the technique to compute lattice integrals is described. In appendix A the analytical expressions and numerical values of one-loop diagrams are presented and in appendix B the method for subtracting infrared divergences is briefly explained.

2 The effective theory for slow heavy quarks

In this section we review the basic results and formulas of the $SHQET$. The Georgi lagrangian describing a heavy quark Q with velocity $v^\mu = (\sqrt{1+\vec{v}^2}, \vec{v})$ in Minkowsky space [2]

$$\mathcal{L}(x) \;=\; Q^\dagger(x) iv \cdot D(x) Q(x) \tag{6}$$

is decomposed as

$$\mathcal{L}(x) \;=\; \mathcal{L}_0(x) \;+\; \mathcal{L}_I(x) \tag{7}$$

where

$$\mathcal{L}_0(x) \;=\; Q^\dagger(x) i D_0(x) Q(x) \tag{8}$$

is the static unperturbed lagrangian and

$$\mathcal{L}_I(x) \;=\; Q^\dagger(x) i [D_0(v_0-1) - \vec{v}\cdot\vec{D}] Q(x) \tag{9}$$

is a perturbation lagrangian giving rise to the motion of Q.

From this splitting it is easy to derive the following expansion of the propagator of Q [17]

$$\begin{aligned} S(x,y;v) \;&=\; -i\,\Theta(t_x-t_y)\Bigg(P(t_x,t_y) + \int_{t_y}^{t_x} dt_z P(t_x,t_z)\vec{v}\cdot\vec{D}(\vec{x},t_z)P(t_z,t_y) \\ &+\; \int_{t_y}^{t_x} dt_z \int_{t_y}^{t_z} dt_w P(t_x,t_z)\vec{v}\cdot\vec{D}(\vec{x},t_z)P(t_z,t_w)\vec{v}\cdot\vec{D}(\vec{x},t_w)P(t_w,t_y) \\ &\qquad -\frac{v^2}{2}P(t_x,t_y) + \ldots \Bigg)\,\delta(\vec{x}-\vec{y}) \end{aligned} \tag{10}$$

where $P(t_b,t_a)$ is a P-line in the time direction joining the point $(\vec{x},t_a)$ with the point $(\vec{x},t_b)$

$$P(t_b,t_a) \;=\; P\exp\left(ig\int_{t_a}^{t_b} A_0(\vec{x},s)ds\right) \tag{11}$$

$S(x,y;v)$ is expressed as a sum of static propagators with an increasing number of local insertions of $(\vec{v}\cdot\vec{D})$ giving rise to the 'perturbative' motion of Q.

The spin structure of Q is taken into account multiplying $S(x,y;v)$ by $(1+\not{v})/2$

$$\begin{aligned} H(v) \;&=\; \frac{1+\not{v}}{2}\,S(v) \\ &=\; \frac{1+\gamma_0}{2}S^{(0)} + \left(\frac{1+\gamma_0}{2}S^{(1)} - \frac{\gamma_3}{2}S^{(0)}\right)v_3 \\ &+\; \left(\frac{1+\gamma_0}{2}S^{(2)} - \frac{\gamma_3}{2}S^{(1)} + \frac{\gamma_0}{4}S^{(0)}\right)v_3^2 \;+\; O(v_3^3) \end{aligned} \tag{12}$$

where we have taken the heavy-quark moving along the z-axis. Inserting the propagator $H(v)$ in the Green's functions describing the dynamics of heavy flavored hadrons, we have the following expansion in powers of v_3

$$G(v) \;=\; G^{(0)} \;+\; G^{(1)}\,v_3 \;+\; G^{(2)}\,v_3^2 + \cdots \tag{13}$$

Consider now the following three- and two-point correlation functions

$$C_3(t,t') \;=\; \int d^3x d^3x' \langle 0 \mid T\;[O_D^\dagger(x'), J_\mu^{b\to c}(x), O_B(0)] \mid 0\rangle \tag{14}$$

$$C_B(t) \;=\; \int d^3x \langle 0 \mid T\;[O_B^\dagger(x), O_B(0)] \mid 0\rangle \tag{15}$$

$$C_D(t'-t) \;=\; \int d^3x' \langle 0 \mid T\;[O_D^\dagger(x'), O_D(x)] \mid 0\rangle \tag{16}$$

where $O_H(x)$ is an interpolating field for the H meson. The simplest choice (which we adopt in the following) is: $O_H(x) = \overline{Q}(x) i\gamma_5 q(x)$, where $Q(x) = b(x),\ c(x)$ for B and D mesons respectively, and $q(x)$ is a light quark field. For large euclidean times, $t \to \infty$, $t' - t \to \infty$, the matrix element (3) is given by

$$\langle D, v \mid J_\mu^{b\to c}(0) \mid B, v' \rangle \;=\; \sqrt{Z_B\ Z_D}\ \frac{C_3(t,t')}{C_B(t)\ C_D(t'-t)} \tag{17}$$

where Z_B and Z_D are the renormalization constants of the operators $O_B(x)$ and $O_D(x)$, given by

$$\begin{aligned}\sqrt{Z_B} &= \langle B, v' \mid O_B(0) \mid 0 \rangle \\ \sqrt{Z_D} &= \langle D, v \mid O_D(0) \mid 0 \rangle\end{aligned} \tag{18}$$

Since both the wave functions and the interpolating fields in eqs.(18) are pseudoscalars, the matrix elements do not depend on the velocity $v(v')$, unless we deal with smeared currents [11].

Inserting the propagator (12) for the c quark in $C_3(t,t')$ and $C_D(t'-t)$ we derive expansions of the form (13). Inserting them into eq.(17), we get the following expression for the derivatives of the Isgur-Wise function with respect to v_4 at the zero recoil point $v_4 = 1$

$$\left[\frac{C_3^{(2)}}{C_3^{(0)}} - \frac{C_D^{(2)}}{C_D^{(0)}}\right] = \frac{1}{2}(\xi^{(1)}(1) + \frac{1}{2}) \tag{19}$$

$$\left[\frac{C_3^{(4)}}{C_3^{(0)}} - \frac{C_D^{(4)}}{C_D^{(0)}} + \left(\frac{C_D^{(2)}}{C_D^{(0)}}\right)^2 - \frac{C_D^{(2)}}{C_D^{(0)}}\,\frac{C_3^{(2)}}{C_3^{(0)}}\right] = \frac{1}{4}(\xi^{(2)}(1) - \frac{1}{2}) \tag{20}$$

where we have used the identity

$$\frac{C_3^{(0)}}{C_B\ C_D^{(0)}} = \sqrt{\frac{2M_D\ 2M_B}{Z_D\ Z_B}} \tag{21}$$

Higher derivatives can be computed similarly.

3 Lattice regularization

We consider the discretization of the effective theory for heavy quarks proposed in ref.[14], forward in time and symmetric in space. For a motion of Q with velocity along the z-axis $v^\mu = (0, 0, v_3, \sqrt{1 + v_3^2})$, the action iS is given by

$$\begin{aligned}iS &= -\sum_x\ v_4 \psi^\dagger(x)\ [\psi(x) - U_4^\dagger(x)\psi(x - \vec{4})] + \\ &\quad -i\frac{v_3}{2}\psi^\dagger(x)\ [U_3(x + \vec{3})\psi(x + \vec{3}) - U_3^\dagger(x)\psi(x - \vec{3})]\end{aligned} \tag{22}$$

where $\vec{\mu}$ is a unit vector in the direction μ, and $U_\mu(x)$ are the links related to the gauge field by $U_\mu(x) \;=\; \exp[-igA_\mu(x-\vec{\mu}/2)]$.

The Feynman rules are those of the static theory plus additional interactions generating the motion of Q. Assuming a convention for the Fourier transform according to which $\psi(x) \sim \exp(ik{\cdot}x)$, we have

$$
\begin{aligned}
iS^{(0)}(k) &= \frac{1}{1-e^{-ik_4}+\epsilon} && (23)\\
V^{(0)}_\mu &= i\,g\,\delta_{\mu 4}\,t_a\;e^{-i(k_4+k'_4)/2} && (24)\\
V^{(0)\ tad}_{\mu\nu} &= -\frac{g^2}{2}\,\delta_{\mu 4}\delta_{\nu 4}\,t_a t_b\;e^{-ik_4} && (25)
\end{aligned}
$$

The linear interactions in v_3 are given by

$$
\begin{aligned}
V^{(1)} &= -v_3\,\sin k_3 && (26)\\
V^{(1)}_\mu &= g\,v_3\,\delta_{\mu 3}\,t_a\;\cos(k_3+k'_3)/2 && (27)\\
V^{(1)\ tad}_{\mu\nu} &= \frac{g^2\,v_3}{2}\,\delta_{\mu 3}\delta_{\nu 3}\,t_a t_b\;\sin k_3 && (28)
\end{aligned}
$$

and the linear interactions in (v_4-1) are given by

$$
\begin{aligned}
V^{(2)} &= -(v_4-1)(1-e^{-ik_4}) && (29)\\
V^{(2)}_\mu &= i\,g\,(v_4-1)\,\delta_{\mu 4}\,t_a\,e^{-i(k_4+k'_4)/2} && (30)\\
V^{(2)\ tad}_{\mu\nu} &= -\frac{g^2}{2}\,(v_4-1)\,\delta_{\mu 4}\delta_{\nu 4}\,t_a t_b\,e^{-ik_4} && (31)
\end{aligned}
$$

where k and k' denote respectively the momenta of the incoming and outgoing heavy quark, and V_μ is the interaction vertex of the heavy quark with a gluon provided with a polarization along the μ axis. $V^{tad}_{\mu\nu}$ are the vertices for the emission of two gluons, for the case of the tadpole graph $(k=k')$. We notice that the vertices labelled $V^{(2)}$ contain second and higher orders in the velocity v_3. It is convenient to keep them unexpanded. Finally, note that the conventions for the sign of the Fourier transform and of the velocity are not independent, if one wishes to intend k as the residual momentum of the heavy quark.

4 Renormalization of lattice operators

Since the lattice effective theory and the continuum one are two different versions of the same physical theory, the matrix elements computed in both theories must coincide. This is a non-trivial condition to impose (matching condition). We match amplitudes of the bare lattice theory onto the corresponding ones of the continuum theory renormalized in some chosen scheme (such as for example $\overline{MS}$). If the lattice lagrangian and the continuum one have at the beginning the same parameters (masses, couplings, etc.), matching is accomplished adding appropriate counterterms to the lattice lagrangian. If we are interested also in the matrix elements of composite operators, an analogous matching has to be performed: appropriate counterterms have to be added to the lattice composite operators. Because of mixing, to a renormalized operator in the continuum it corresponds in general a linear combination of lattice bare operators.

The technique for obtaining the lattice counterpart of a continuum operator is standard [20, 21]. In lattice regularization the inverse of the lattice spacing $1/a$ acts as an ultraviolet cut-off, and bare lattice amplitudes depend explicitely on a. Continuum amplitudes depend instead on a renormalization point μ. To avoid large logarithms in the matching constants ($\log a\mu >> 1$), let us first match the amplitudes by taking $\mu = a^{-1}$. At this stage we are therefore dealing with the finite discrepancies coming from the use of different regulators. The relation between continuum and lattice operators at one-loop level for $\mu = a^{-1}$ is given by

$$O_i^{Cont}(\mu = a^{-1}) \; = \sum_j \left[\, \delta_{ij} \, + \, \left(\frac{\alpha_s(a^{-1})}{\pi}\right) \, \delta Z_{ij} \,\right] O_j^{Latt}(a) \tag{32}$$

where O_i^{Cont} are the operators in the continuum we are interested in, O_j^{Latt} are the lattice ones and δZ_{ij} are finite renormalization (or mixing) constants. The sum extend over all the operators that can mix with O_i^{Latt} as a consequence of the symmetry breaking induced by the continuum and the lattice regularization procedure. By sandwiching the operators between arbitrary external states of momenta p, we derive the following relation involving Green's functions of the bare lattice operators or the (renormalized) continuum ones

$$< \, O_i^{Cont,Latt} \; >= \sum_j \left[\, \delta_{ij} \, + \, \left(\frac{\alpha_s(a^{-1})}{\pi}\right) \, C_{ij}^{Cont,Latt}(p) \,\right] \, < \, O_j \, >^{(0)} \tag{33}$$

where the superscript (0) denotes tree level matrix elements. Demanding compatibility between (32) and (33), we derive

$$\delta Z_{ij} \; = \; \lim_{a \to 0} \left[\, C_{ij}^{Cont}(p) \; - \; C_{ij}^{Latt}(p) \, \right] \tag{34}$$

The mixing coefficients δZ_{ij} are independent of both the external states and the momentum configuration used to calculate matrix elements of O_i.

We consider now the matching in the more general case $\mu a \neq 1$. Since we already matched the amplitudes at $\mu a = 1$, we need only to evolve the mixing coefficients δZ_{ij} from $\mu = 1/a$ to a generic renormalization point with RG techniques. At one loop-level, the δZ_{ij}'s do depend on the renormalization scheme. In order to obtain a renormalization scheme independent matching condition, the two-loop anomalous dimension contribution must be taken into account in the diagonal terms [22, 23]

$$
\begin{aligned}
O_i^{Cont}(\mu) &= \left(\frac{\alpha_s(a^{-1})}{\alpha_s(\mu)}\right)^{\gamma_1/\beta_1} \left[\, 1 + \left(\frac{\alpha_s(a^{-1})}{\pi} - \frac{\alpha_s(\mu)}{\pi}\right) R_{O_i} \,\right] \\
&\times \sum_j \left[\, \delta_{ij} + \left(\frac{\alpha_s(a^{-1})}{\pi}\right) \delta Z_{ij} \,\right] O_j^{Latt}(a)
\end{aligned} \tag{35}
$$

where

$$
R_{O_i} = \frac{1}{\beta_1^2} \left[\, \gamma_2 \, \beta_1 - \gamma_1 \, \beta_2 \,\right] \tag{36}
$$

with γ_n the n-loop anomalous dimension of the operator O_i defined by

$$
\begin{aligned}
O_i^R &= Z_O \, O_i^B \\
\gamma &= -\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \log Z_O = \gamma_1 \left(\frac{\alpha_s}{\pi}\right) + \gamma_2 \left(\frac{\alpha_s}{\pi}\right)^2 + \cdots
\end{aligned} \tag{37}
$$

and with β_1 and β_2 the one-loop and two-loop coefficients of the β-function respectively

$$
\beta(\alpha_s) = \beta_1 \frac{\alpha_s}{\pi} + \beta_2 \left(\frac{\alpha_s}{\pi}\right)^2 + \ldots \tag{38}
$$

We have

$$
\begin{aligned}
\beta_1 &= -\frac{11}{2} + \frac{1}{3} n_F \\
\beta_2 &= -\frac{51}{4} + \frac{19}{12} n_F
\end{aligned} \tag{39}
$$

The expression for the running coupling constant is given by

$$
\alpha_s(\mu) = \frac{2\pi}{-\beta_1 \, \log(\mu^2/\Lambda^2)} \left[\, 1 + \frac{2\beta_2 \, \log \, \log(\mu^2/\Lambda^2)}{\beta_1^2 \, \log(\mu^2/\Lambda^2)} \,\right] \tag{40}
$$

where n_F is the number of active quark flavors and we can take $\Lambda = 200$ MeV in the $\overline{MS}$ scheme.

It is expected that for the values of a^{-1} currently used in lattice simulations the matching should depend only weakly on the continuum regularization. This is why physicists usually compute matching constants without including the two-loop anomalous dimension.

Finally, let us briefly expose the problem of the power divergences in lattice computations. Since QCD is asymptotically free, the matching constants δZ_{ij} can in principle be computed with RG-improved perturbation theory in the limit $a \to 0$ (in practise one requires $a\Lambda \ll 1$). Unfortunately, the mixing coefficients contain in some cases inverse powers of a, which divergence as a goes to zero. Then, in computations of matrix elements of the continuum operator, the leading term is this mixing term of $O(1/a^n)$ which is a lattice artifact generated by the regularization procedure and thus must be subtracted. In other words, in order to obtain finite Green functions of composite operators on the lattice, we must subtract power divergences in a^{-1} from the Monte Carlo data. It has been argued that it can be done in perturbation theory. However, as pointed out in refs.[18, 19, 24, 25], it is not clear that the coefficients of power divergences can be calculated to sufficient accuracy in perturbation theory. In general, very difficult non-perturbative subtractions for lattice Green's functions are required.

5 Renormalization of the lattice $SHQET$

In this section we discuss the renormalization of the $SHQET$ given by the lagrangian (22), i.e. the determination of the counterterms which have to be introduced to match amplitudes of the lattice $SHQET$ onto those of the continuum $HQET$.

To obtain the renormalized operator $(\vec{v} \cdot \vec{D})$ we compute the one-loop heavy quark self-energy with insertions of $(\vec{v} \cdot \vec{D})$ using the lattice Feynman rules of section 3. This is equivalent to calculate the one-loop heavy quark self-energy up to a given order in the velocity v_3.

After that, we match the heavy quark propagator of the lattice $SHQET$ onto the continuum $HQET$ propagator, expanded in v_3.

For calculational convenience, we will take equal incoming and outgoing momenta and adopt the Feynman gauge for the gluon propagator. The infrared divergences which appear at zero external momenta are regulated giving the gluon a fictitious mass λ. No problem arises with non-abelian gauge symmetry because all the amplitudes are QED-like. Other choices are possible for the infrared regulator, such as for example to take virtual external states [23]. However, by using a non-vanishing gluon mass, we achieve a great simplification in computing the lattice loop integrals. Indeed, we can safely Taylor expand the corresponding diagrams about zero external momenta up to order $O(a)$ to determine all nonvanishing terms as a goes to zero. Upon doing this, we will subtract the infrared (logarithmic) divergences from the integrals with the technique explained in Appendix B.

The computation of the diagrams will be done with two different methods for dealing with the non-covariant poles coming from static lines. The first method is based on partial integration with respect to k_4 (the fourth-component of the euclidean loop momentum) in order to eliminate the non-covariant poles. The integrand is reduced to a covariant form and can be computed with usual techniques. The second method is to integrate analitically over k_4 using the ϵ-prescription of the static heavy-quark propagator and the Cauchy's theorem. This latter technique involves less algebra, but leads to non-covariant 3-dimensional integrals. The comparison of the results obtained with the two methods provides us with a check not only of our analytical computation but also of our numerical calculations.

5.1 Heavy Quark Self-Energy up to $O(v_3^2)$

To illustrate the method of partial integration, in this section we briefly describe the computation of the diagrams that determine the heavy-quark self-energy up to $O(v_3^2)$, which are depicted in Fig.1. We do not consider the insertion of v_4-vertices (i.e. those in eqs.(29) to (31)) because their contribution can be shown to be trivial. We will treat this subject in detail in the next section.

We start by computing the diagrams with one insertion of the operator $(\vec{v}\cdot\vec{D})$ in Fig.1. Diagrams A.1 and A.2 vanish in the Feynman gauge. This happens because the gluon is emitted by the operator $(\vec{v}\cdot\vec{D})$ with a polarization along the z axis, while it is absorbed by the static vertex with a polarization along the time axis. The (amputated) amplitude of diagram $A.3$ is given by

$$A_3(p) \;=\; g^2\, C_F\, v_3\, e^{-ip_4} \int_{-\pi}^{+\pi} \frac{d^4k}{(2\pi)^4} \frac{\sin k_3\, e^{-ik_4}}{(1-e^{ik_4}+i\epsilon)^2}\, \frac{1}{2\Delta_1(k-p)} \tag{41}$$

where p is the external momentum, $C_F = \sum t_a t_a = (N^2-1)/2N$ for an $SU(N)$ gauge theory, and $\Delta_1(l) = \sum_{\mu=1}^{4} 1 - \cos l_\mu + (a\lambda)^2/2$.

Now, $A_3(p)$ vanishes at zero external momentum $p = 0$,

$$A_3(p=0) \;=\; 0, \tag{42}$$

because the integrand is odd in k_3.

First derivatives of $A_3(p)$ with respect to the external momentum contain logarithmic ultraviolet divergences. The only non-vanishing derivative is that one with respect to p_3. With a partial integration with respect to k_4 of the factor

$$\frac{e^{-ik_4}}{(1-e^{-ik_4}+\epsilon)^2} \tag{43}$$

we reduce the integral to the following form

$$\left(\frac{\partial A_3}{\partial p_3}\right)_0 = \frac{g^2 C_F}{16\pi^2}\, v_3\, \frac{1}{6\pi^2}\int_{-\pi}^{+\pi} \mathrm{d^4k}\; \frac{\eta(\vec{\mathrm{k}})\ (1+\cos \mathrm{k}_4)}{\Delta_1(\mathrm{k})^3} \tag{44}$$

where $\eta(\vec{k}) = \sum_{i=1}^{3} \sin^2 k_i$ and a symmetrization over the spatial momenta has been done. The infrared singularity of the integral is isolated with the technique introduced in ref.[23] and described in detail in ref.[16]. The result can be written as

$$\left(\frac{\partial A_3}{\partial p_3}\right)_0 = \left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\, v_3\, [\ -2\log(a\lambda)^2 + a_3\ ] \tag{45}$$

where the subleading (finite) term a_3 is a constant evaluated numerically, $a_3 = 0.448$.

The tadpole graph $A.4$ is given by

$$A_4(p) = \frac{g^2\, C_F\, v_3}{4}\sin p_3 \int_{-\pi}^{+\pi}\frac{\mathrm{d^4k}}{(2\pi)^4}\frac{1}{\Delta_1(k)} \tag{46}$$

As in the case of diagram $A.3$, $A_4(p)$ vanishes at zero external momentum,

$$A_4(p=0) = 0. \tag{47}$$

The first derivative with respect to p_3 is finite (i.e. does not contain logarithmic divergences) and reads

$$\left(\frac{\partial A_4}{\partial p_3}\right)_0 = \left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\, v_3\ a_4 \tag{48}$$

where a_4 is a numerical constant, $a_4 = 12.23$.

Let us consider now the renormalization of a double insertion of $(\vec{v}\cdot\vec{D})$ at zero momentum. We study the Green function

$$G(z,w) = \int d^4x d^4y\ \langle 0 \mid T[Q(z)\, Q^\dagger(x)(\vec{v}\cdot\vec{D})(x)Q(x)\, Q^\dagger(y)(\vec{v}\cdot\vec{D})(y)Q(y)\, Q^\dagger(w)] \mid 0\rangle \tag{49}$$

The diagrams involved are drawn in Fig.2. Diagrams $B.1$ and $B.2$ vanish in the Feynman gauge. The amplitude of diagram $B.3$ can be written as

$$B_3(p) = -\frac{g^2\, C_F\, v_3^2}{6} e^{-ip_4}\int_{-\pi}^{+\pi}\frac{\mathrm{d^4k}}{(2\pi)^4}\frac{\eta(\vec{k})\ e^{-ik_4}}{(1-e^{-ik_4}+\epsilon)^3\ \Delta_1(p-k)} \tag{50}$$

The amplitude at zero external momentum is given by

$$B_3(0) = \frac{g^2\, C_F}{16\pi^2}\, v_3^2 \frac{-1}{6\pi^2}\int_{-\pi}^{+\pi} \mathrm{d^4k}\ \frac{\eta(\vec{\mathrm{k}})\ \mathrm{e}^{-\mathrm{ik}_4}}{(1-\mathrm{e}^{-\mathrm{ik}_4}+\epsilon)^3\ \Delta_1(\mathrm{k})} \tag{51}$$

It is convenient to reduce the integrand to a covariant form, by eliminating the triple pole coming from the static line. Let us describe in detail the transformation of this integral, which will illustrate the technique to deal with poles of odd order.

We perform first a partial integration with respect to k_4 analogous to that one of $\partial A_3/\partial p_3$. This transformation brings the integral into the form

$$B_3(0) \;=\; \frac{g^2\,C_F}{16\pi^2}\,v_3^2\frac{-1}{24\pi^2}\int_{-\pi}^{+\pi}\mathrm{d}^4\mathrm{k}\frac{\eta(\vec{\mathrm{k}})\;(\mathrm{e}^{\mathrm{i}\mathrm{k}_4}+1)}{(1-\mathrm{e}^{-\mathrm{i}\mathrm{k}_4}+\epsilon)}\frac{1}{\Delta_1(\mathrm{k})^2} \tag{52}$$

The simple non-covariant pole is treated by writing [26]

$$\frac{1}{\Delta_1(k)^2} \;=\; \left(\frac{1}{\Delta_1(k)^2}-\frac{1}{\Delta_1(0,\vec{k})^2}\right)\;+\frac{1}{\Delta_1(0,\vec{k})^2} \tag{53}$$

In the integral containing the difference of gluon propagators,

$$I \;=\; \int_{-\pi}^{+\pi}\mathrm{d}^4\mathrm{k}\;\frac{1+\mathrm{e}^{\mathrm{i}\mathrm{k}_4}}{1-\mathrm{e}^{-\mathrm{i}\mathrm{k}_4}+\epsilon}\;\eta(\vec{\mathrm{k}})\;\left(\frac{1}{\Delta_1(\mathrm{k})^2}-\frac{1}{\Delta_1(0,\vec{\mathrm{k}})^2}\right), \tag{54}$$

one can set $\epsilon=0$. Since the gluon propagator is even with respect to k_4, one can symmetrize the factor

$$\frac{1+e^{ik_4}}{1-e^{-ik_4}}\;\;\rightarrow\;\;1+\cos k_4 \tag{55}$$

The integral I therefore reads

$$I \;=\; \int_{-\pi}^{+\pi}\mathrm{d}^4\mathrm{k}\;\frac{(1+\cos\mathrm{k}_4)\;\eta(\vec{\mathrm{k}})}{\Delta_1(\mathrm{k})}\;\;-2\pi\int_{-\pi}^{+\pi}\mathrm{d}^3\mathrm{k}\;\frac{\eta(\vec{\mathrm{k}})}{\Delta_1(0,\vec{\mathrm{k}})} \tag{56}$$

In the remaining integral

$$J \;=\; \int_{-\pi}^{+\pi}\mathrm{d}\mathrm{k}_4\frac{1+\mathrm{e}^{\mathrm{i}\mathrm{k}_4}}{1-\mathrm{e}^{-\mathrm{i}\mathrm{k}_4}+\epsilon}\;\int_{-\pi}^{+\pi}\mathrm{d}^3\mathrm{k}\frac{\eta(\vec{\mathrm{k}})}{\Delta_1(0,\vec{\mathrm{k}})^2}, \tag{57}$$

we perform the contour integration over k_4 analytically by setting $z=\exp(ik_4)$. $B_3(0)$ is finally expressed as a sum of a 4-dimensional integral and a 3-dimensional one

$$B_3(0) \;=\; \frac{g^2C_F}{16\pi^2}v_3^2\left(-\frac{1}{24\pi^2}\int d^4k\frac{(1+\cos k_4)\;\eta(\vec{k})}{\Delta_1(k)^2}-\frac{1}{12\pi}\int d^3k\frac{\eta(\vec{k})}{\Delta_1(0,\vec{k})^2}\right) \tag{58}$$

The integrals in eq.(58) are infrared finite and are easily computed numerically

$$B_3(0) \;=\; \left(\frac{\alpha_s}{\pi}\right)\;\frac{C_F}{4}\,v_3^2\;b_{30} \tag{59}$$

where $b_{30}=-5.044$.

The first derivative of $B_3(p)$ with respect to p_4 is logarithmically divergent, and is given by

$$\left(\frac{\partial B_3}{\partial p_4}\right)_0 \;=\; -iA_3(0)-\frac{g^2\,C_F\,v_3^2}{6}\int\frac{d^4k}{(2\pi)^4}\frac{\sin k_4\;e^{-ik_4}}{(1-e^{-ik_4}+\epsilon)^3}\frac{\eta(\vec{k})}{\Delta_1(k)^2} \tag{60}$$

Using the same tricks as for $B_3(0)$, this integral is transformed into

$$\begin{aligned}\left(\frac{\partial B_3}{\partial p_4}\right)_0 &= -iB_3(0) + \frac{g^2 C_F}{16\pi^2} v_3^2 \Bigg(\frac{i}{12\pi^2} \int d^4k \frac{\eta(\vec{k})}{\Delta_1(k)^3} [1 + 2\cos k_4 + \cos 2k_4] \\ &+ \frac{i}{12\pi^2} \int d^4k \frac{\eta(\vec{k})}{\Delta_1(k)^2} [1/2 + \cos k_4] + \frac{i}{12\pi} \int d^3k \frac{\eta(\vec{k})}{\Delta_1(0,\vec{k})^2} \Bigg) \end{aligned} \tag{61}$$

The logarithmic divergence of the amplitude (the $\log(a\lambda)$ term) is entirely contained in the first integral. The computation yields

$$\left(\frac{\partial B_3}{\partial p_4}\right)_0 = i \left(\frac{\alpha_s}{\pi}\right) \frac{C_F}{4} v_3^2 \, [\, -b_{30} - 2\log(a\lambda)^2 + b_{31} \,] \tag{62}$$

where $b_{31} = 4.988$.

Finally, the amplitude of diagram B_4 is

$$B_4(p) = \frac{g^2 C_F \, v_3^2}{12} \int_{-\pi}^{+\pi} \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4} \frac{3 + \sigma(\vec{k})}{1 - e^{-ik_4} + \epsilon} \frac{1}{\Delta_1(k-p)} \tag{63}$$

where $\sigma(\vec{k}) = \sum_{i=1}^3 \cos k_i$.

The computation of $B_4(p)$ is analogous to that of $B_3(p)$. We have

$$B_4(0) = \frac{g^2 C_F}{16\pi^2} v^2 \left(\frac{1}{24\pi^2} \int \mathrm{d}^4\mathrm{k} \, \frac{3 + \sigma(\vec{\mathrm{k}})}{\Delta_1(\mathrm{k})} + \frac{1}{12\pi} \int \mathrm{d}^3\mathrm{k} \, \frac{3 + \sigma(\vec{\mathrm{k}})}{\Delta_1(0, \vec{\mathrm{k}})} \right) \tag{64}$$

Upon a numerical computation we find

$$B_4(0) = \left(\frac{\alpha_s}{\pi}\right) \frac{C_F}{4} v_3^2 \, b_{40} \tag{65}$$

where $b_{40} = 20.566$.

The derivative with respect to p_4 reads

$$\left(\frac{\partial B_4}{\partial p_4}\right)_0 = \frac{g^2 C_F}{16\pi^2} v_3^2 \frac{-i}{24\pi^2} \int_{-\pi}^{+\pi} \mathrm{d}^4\mathrm{k} \frac{[\, 3 + \sigma(\vec{\mathrm{k}}) \,][\, 1 + \cos \mathrm{k}_4]}{\Delta_1(\mathrm{k})^2} \tag{66}$$

and the corresponding numerical computation yields

$$\left(\frac{\partial B_4}{\partial p_4}\right)_0 = i \left(\frac{\alpha_s}{\pi}\right) \frac{C_F}{4} v_3^2 \, [\, 2\log(a\lambda)^2 + b_{41} \,] \tag{67}$$

where $b_{41} = -2.485$.

Putting all contributions together, we can write the heavy-quark self-energy up to order $O(v_3^2)$ as

$$\begin{aligned}\Sigma(p,v) &= \frac{1}{a} \left(\frac{\alpha_s}{\pi}\right) \frac{C_F}{4} \left[\Sigma_0^{(0)} + v_3^2 \, \Sigma_0^{(2)} \right] v_4 \\ &+ \left(\frac{\alpha_s}{\pi}\right) \frac{C_F}{4} \left[\Sigma_{40}^{(0)} + v_3^2 \, \Sigma_{40}^{(2)} - 4\log(a\lambda) \right] (i\, p_4 v_4) \\ &+ \left(\frac{\alpha_s}{\pi}\right) \frac{C_F}{4} \left[\Sigma_{30}^{(1)} - 4\log(a\lambda) \right] (p_3 v_3) \\ &+ O(\alpha_s\, a, v_3^3) \end{aligned} \tag{68}$$

where $\Sigma_0^{(0)}$ and $\Sigma_{40}^{(0)}$ are the mass and wave function renormalization of a static heavy quark [26]. Their numerical values are tabulated in Table A.1. On the other hand, the numerical values of the new constants $\Sigma_{0,30,40}^{(1,2)}$ are

$$\begin{aligned}
\Sigma_0^{(2)} &= b_{30} + b_{40} = 15.52 \\
\Sigma_{40}^{(2)} &= b_{31} - b_{30} + b_{41} = 7.55 \\
\Sigma_{30}^{(1)} &= a_3 + a_4 = 12.68
\end{aligned} \tag{69}$$

In the next section we compare (69) with the self-energy calculated by using a different integration technique.

5.2 Heavy Quark Self-Energy beyond $O(v_3^2)$

Here we compute the heavy-quark self-energy at one-loop in the coupling constant α_s but to all orders in the velocity v_3. We will demonstrate that the discretized lagrangian (22) reproduces the correct infrared behaviour of the HQET to all orders in the velocity, as it should be. The calculation will be performed utilizing a different method from the one used in the previous section. This is useful to check both our analytical and numerical results.

We start by noting that at one-loop the diagrams that contribute to the self-energy of the heavy quark at order $O(v_3^m)$ are those depicted in Fig.3. In fact, these diagrams represent the only two ways of inserting m v_3-vertices (i.e. those in eqs.(26) to (28)) into the gluon-loop self-energy diagram. The reader may however argue that we are ignoring the considerable number of v_4-vertex insertions (i.e. those in eqs.(29) to (31)) which give rise to corrections to the heavy-quark self-energy of the same order in the velocity as those considered above (see Fig.4). This is of course true, but it is very easy to show that the full effect of all possible v_4-vertex insertions into the diagrams of Fig.3 is just to multiply them by a factor $(1/v_4)^{m-1}$.

To demonstrate this result, consider the diagrams of Fig.3. We can insert n v_4-vertices only in two ways, namely, A: one at any of the quark-gluon vertices and the remaining $n-1$ on the heavy-quark line and B: all n insertions into the heavy quark propagator inside the gluon loop. The resulting diagrams are shown in Fig.4. Now, the effect of a v_4-vertex insertion into a heavy quark propagator is simply to multiply the same propagator by $(-)\,(v_4-1)$, as it can be seen from the Feynman rule in (29). If the insertion is at a quark-gluon vertex, the effect is to multiply the same vertex by (v_4-1) (see eq.(30)). For example, two v_4-vertex insertions, one at the quark-gluon vertex and the other into the heavy-quark propagator inside the gluon loop, give $(-)\,(v_4-1)^2$ times the old diagram without any v_4-vertex insertions. Therefore, we only have to count the number of topologically different diagrams with n v_4-vertex insertions in each class of Fig.4, for all diagrams in a class give the same contribution to the heavy-quark self-energy. To this end, we observe that the number of different ways we can insert n v_4-vertices on a heavy-quark line where there are m v_3-vertices is $(n+m)!/[n!\,m!]$ (the old combinatorial problem of distributing n balls in $m+1$ boxes). Therefore, the sum of the graphs in Fig.4 gives

$$\begin{aligned}
\text{Fig.4} \;&=\; (-)^n \frac{(n+m-2)!}{[n!\;(m-2)!]}\,(v_4-1)^n\,\text{C.1} \,+\, (-)^n \frac{(n+m)!}{[n!\;m!]}\,(v_4-1)^n\,\text{C.2} \\
&+\; 2\,(-)^{n-1} \frac{(n-1+m)!}{[(n-1)!\;m!]}\,(v_4-1)^n\,\text{C.2} \,+\, (-)^{n-2} \frac{(n-2+m)!}{[(n-2)!\;m!]}\,(v_4-1)^n\,\text{C.2} \\
&=\; (-)^n \frac{(n+m-2)!}{[n!\;(m-2)!]}\,(v_4-1)^n\,(\text{C.1} \,+\, \text{C.2})
\end{aligned} \tag{70}$$

Summing from $n=0$ to ∞, we get

$$\sum_{n=0}^{\infty} \text{Fig.4} = \frac{1}{v_4^{m-1}}\,(\text{C.1} \,+\, \text{C.2}) \tag{71}$$

as anticipated.

We turn now to the computation of diagrams of Fig.3. Their amplitudes are

$$\begin{aligned}
C_1(m,p) \;&=\; \frac{-1}{a}\left(\frac{\alpha_s}{\pi}\right) C_F\,(-v_3)^m\, e^{-2ip_4}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d^3k}{(2\pi)^3}\,\sin^m(p_3-k_3) \\
&\times\; \oint \frac{dz}{2\pi i\, z}\,\frac{z}{[1\,+\,\epsilon\,-\,e^{-ip_4}\,z]^{m+1}}\,\frac{-z}{(z-z_-)(z-z_+)}
\end{aligned} \tag{72}$$

and

$$\begin{aligned}
C_2(m,p) \;&=\; \frac{1}{a}\left(\frac{\alpha_s}{\pi}\right) C_F\,(-v_3)^m\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d^3k}{(2\pi)^3}\,\sin^{m-2}(p_3-k_3)\,\cos^2(p_3-k_3/2) \\
&\times\; \oint \frac{dz}{2\pi i\, z}\,\frac{z}{[1\,+\,\epsilon\,-\,e^{-ip_4}\,z]^{m-1}}\,\frac{-z}{(z-z_-)(z-z_+)}
\end{aligned} \tag{73}$$

where $z\,=\,e^{ik_4}$, the contour integral is along the unit circle and we have used the fact that the gluon propagator can be written as

$$\frac{1}{\sum_{\mu=1}^{4}(1-\cos(k_\mu))\,+\,(a\lambda)^2/2} \;=\; \frac{-z}{(z-z_-)(z-z_+)} \tag{74}$$

with $z_\pm$ being the solutions of $z_\pm^2\,-\,2\,(1+B)\,z_\pm\,+\,1\,=\,0$ and

$$B = \sum_{\mu=1}^{3}(1-\cos(k_\mu))\,+\,(a\lambda)^2/2 \tag{75}$$

The non-vanishing terms as a goes to zero are $C_{1,2}(p=0)$, which contain a linear divergence, and the first derivatives of $C_{1,2}(p)$ with respect to p_3 and p_4 at $p=0$, which are logarithmically divergent. In either case, the calculation reduces to the computation of the contour integral over z which can easily be performed taking into account that only the pole $z=z_-$ lies in the unit circle. Furthermore, the ϵ-prescription tells us that the pole of the quark propagator does not contribute to the contour integral when the Cauchy's theorem is used. The final result is, for the non-derivative contribution,

$$
\begin{aligned}
C_1(2m+1,p=0) &= 0 \quad \text{by parity} \\
C_1(2m,p=0) &= \frac{-1}{a}\left(\frac{\alpha_s}{\pi}\right) C_F\, v_3^{2m}\, \frac{2}{4^{m+1}} \left[\mathrm{Si}^{(20)}(m-1) + \mathrm{Si}^{(11)}(m-1)\right] \\
C_2(2m+1,p=0) &= 0 \quad \text{by parity} \\
C_2(2m,p=0) &= \frac{1}{a}\left(\frac{\alpha_s}{\pi}\right) C_F\, v_3^{2m}\, \frac{1}{4^{m}} \left[\mathrm{Cs}^{(10)}(m-1) + \mathrm{Cs}^{(01)}(m-1)\right]
\end{aligned}
\tag{76}
$$

The derivative with respect to p_3 at $p=0$ gives

$$
\begin{aligned}
\left(\frac{\partial C_1(2m,p)}{\partial p_3}\right)_0 &= 0 \quad \text{by parity} \\
\left(\frac{\partial C_1(2m-1,p)}{\partial p_3}\right)_0 &= \left(\frac{\alpha_s}{\pi}\right) C_F\, v_3^{2m-1}\, \frac{2m-1}{4^{m}} \left[2\,\mathrm{Cs}^{(11)}(m-1) - \mathrm{Id}^{(11)}(m-1)\right] \\
\left(\frac{\partial C_2(2m,p)}{\partial p_3}\right)_0 &= 0 \quad \text{by parity} \\
\left(\frac{\partial C_2(2m+1,p)}{\partial p_3}\right)_0 &= \left(\frac{\alpha_s}{\pi}\right) C_F\, v_3^{2m+1}\, \frac{1}{4^{m}} \\
&\times \left[\frac{2m+1}{2}\left\{\mathrm{Si}^{(10)}(m-1) + \mathrm{Si}^{(01)}(m-1) + \mathrm{Si}^{(11)}(m-1)\right\}\right. \\
&- \left.(2m-1)\left\{\mathrm{Cs}^{(10)}(m-1) + \mathrm{Cs}^{(01)}(m-1) + \mathrm{Cs}^{(11)}(m-1)\right\}\right]
\end{aligned}
\tag{77}
$$

Finally, the derivatives with respect to p_4 at $p=0$ yield

$$
\begin{aligned}
\left(\frac{\partial C_1(2m+1,p)}{\partial p_4}\right)_0 &= 0 \quad \text{by parity} \\
\left(\frac{\partial C_1(2m,p)}{\partial p_4}\right)_0 &= i\left(\frac{\alpha_s}{\pi}\right) C_F\, v_3^{2m}\, \frac{1}{4^{m}} \\
&\times \left[\mathrm{Si}^{(20)}(m-1) + \mathrm{Si}^{(11)}(m-1) + \frac{2m+1}{2}\,\mathrm{Si}^{(21)}(m-1)\right] \\
\left(\frac{\partial C_2(2m+1,p)}{\partial p_4}\right)_0 &= 0 \quad \text{by parity} \\
\left(\frac{\partial C_2(2m,p)}{\partial p_4}\right)_0 &= -i\left(\frac{\alpha_s}{\pi}\right) C_F\, v_3^{2m}\, \frac{2m-1}{4^{m}}\,\mathrm{Cs}^{(11)}(m-1)
\end{aligned}
\tag{78}
$$

where $\mathrm{Si}^{(\alpha\beta)}(m)$, $\mathrm{Cs}^{(\alpha\beta)}(m)$ and $\mathrm{Id}^{(\alpha\beta)}(m)$ are three-dimensional integrals which analytical expressions and numerical values for several m can be found in appendix A and Table A.1 respectively.

We are now in a position to giving the expression of the heavy-quark self-energy on the lattice at any order in the velocity v_3. In fact, it can be written as

$$
\begin{aligned}
\Sigma(p,v) &= \frac{1}{a}\left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\; v_4 \sum_{i=0}^{\infty}\left(\frac{v_3}{v_4}\right)^{2i}\Sigma_0^{(2i)} \\
&+ \left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\left[\sum_{i=0}^{\infty}\left(\frac{v_3}{v_4}\right)^{2i}\left\{\Sigma_{40}^{(2i)} - \Sigma_{41}^{(2i)}\log(a\lambda)\right\}\right](ip_4 v_4) \\
&+ \left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\left[\sum_{i=0}^{\infty}\left(\frac{v_3}{v_4}\right)^{2i}\left\{\Sigma_{30}^{(2i+1)} - \Sigma_{31}^{(2i+1)}\log(a\lambda)\right\}\right](p_3 v_3) \\
&+ O(\alpha_s\, a)
\end{aligned}
\tag{79}
$$

where the constants $\Sigma_{0,30,40}^{(m)}$ can easily be obtained from the results in eq.(76) to (78). Their analytical expressions and numerical values are listed in appendix A and Table A.2.

To finish this section, we wish to discuss in detail the values of constants $\Sigma_{41}^{(m)}$ and $\Sigma_{31}^{(m)}$ which determine both the one-loop wave function renormalization of the heavy-quark with velocity v_3 and the anomalous dimension of the operator $(\vec{v}\cdot\vec{D})$.

We begin with $\Sigma_{41}^{(2m)}$. For $m = 0$, it is the static heavy-quark wave-function renormalization, first computed in ref.[26]. We reproduce their result $\Sigma_{41}^{(0)} = 4$ (see eq.(A.6) and Table A.1). If $m > 0$, $\Sigma_{41}^{(2m)}$ is determined by the pole part of the sum of the derivatives with respect to p_4 of diagrams C.1 and C.2. The terms that contain a logarithmic divergence are $\mathrm{Si}^{(21)}(m)$ and $\mathrm{Cs}^{(11)}(m)$ which appear in $\Sigma_{41}^{(2m)}$ through the combination

$$
\Sigma_{41}^{(2m)} \propto \frac{2m+1}{2}\,\mathrm{Si}^{(21)}(m)\,|_{\mathrm{pole}} - \frac{2m-1}{2}\,\mathrm{Cs}^{(11)}(m)\,|_{\mathrm{pole}} \tag{80}
$$

In appendix B, we calculate the logarithmically divergent part of $\mathrm{Si}^{(21)}(m)$ and $\mathrm{Cs}^{(11)}(m)$. Inserting eqs.(B.10) and (B.11) into (80), we obtain that these integrals conspire order by order in the velocity to yield a vanishing coefficient $\Sigma_{41}^{(2m)}$ for $m > 0$. In other words, the coefficient of the logarithm of the wave-function renormalization of a heavy-quark moving with velocity v_3 is independent of the velocity, as it should be. Moreover, the anomalous dimension turns out to be equal to that of the static theory, which in turn is the same as the one in the continuum. Therefore, we can say that the lagrangian (22) preserves both the infrared and ultraviolet behaviour of the non-expanded theory order by order in the velocity.

Similarly, the pole part of the sum of the derivatives with respect to p_3 of diagrams C.1 and C.2, determine $\Sigma_{31}^{(2m+1)}$. In this case, there is only a term containing a logarithmic divergence,

$\mathrm{Cs}^{(11)}(m)$, which appear in $\Sigma_{31}^{(2m+1)}$ through the combination

$$\Sigma_{31}^{(2m+1)} \propto \frac{2m+1}{4^{m+1}} \left\{ 2\,\mathrm{Cs}^{(11)}(m) \,-\, \mathrm{Id}^{(11)}(m) \right\} |_{\mathrm{pole}} - \frac{2m-1}{4^m}\,\mathrm{Cs}^{(11)}(m-1)\,|_{\mathrm{pole}} \qquad (81)$$

Again, substituting the expressions for the pole parts given in eqs.(B.11) and (B.12) into (81), we observe that the coefficients of the logarithms conspire order by order in the velocity to produce a vanishing $\Sigma_{31}^{(2m+1)}$ for $m > 0$. The only logarithmically divergent term left is that for $m = 0$. Note also that $\Sigma_{31}^{(1)}$ is equal to $\Sigma_{41}^{(0)}$. This fact it very important because it implies that the renormalization constant of the operator $(\vec{v}\cdot\vec{D})$ is finite (see next section), as it should be since this operator is conserved in the static theory. Again, consistency with the non-expanded theory is explicitly shown.

Putting all these things together, we have

$$\begin{aligned}
\Sigma(p,v) \;&=\; \frac{1}{a}\left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\,v_4\sum_{i=0}^{\infty}\left(\frac{v_3}{v_4}\right)^{2i}\Sigma_0^{(2i)} \\
&+\; \left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\left[\sum_{i=0}^{\infty}\left\{\left(\frac{v_3}{v_4}\right)^{2i}\Sigma_{40}^{(2i)}\right\} - 4\,\log(a\lambda)\right](ip_4v_4) \\
&+\; \left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\left[\sum_{i=0}^{\infty}\left\{\left(\frac{v_3}{v_4}\right)^{2i}\Sigma_{30}^{(2i+1)}\right\} - 4\,\log(a\lambda)\right](p_3v_3) \\
&+\; O(\alpha_s\,a)
\end{aligned} \qquad (82)$$

which is one of our most important results. Notice that up to $O(v_3^2)$, the heavy-quark self-energy (82) coincides (within an error of less than 1%) with the one computed in the previous section using a different integration method (see eq.(69)). This fact makes us think that our numerical calculation is correct.

5.3 Wave function and mass renormalization

Having obtained the heavy-quark self-energy, we want to define and compute its wave function renormalization Z_Q (defined by $Q^R = Z_Q^{-1/2}Q^B$) and mass renormalization δM.

In order to get the renormalization constants, we study the heavy-quark propagator near on-shell including order α_s corrections

$$iH(p,v_3) \;=\; \frac{1}{(1-\Sigma_4)\,(ip_4v_4)\,+(1-\Sigma_3)\,(p_3v_3)\,-\,\Sigma_0\,+\,O(p^2)} \qquad (83)$$

where

$$\Sigma_4 \;=\; -i\left(\frac{\partial\Sigma(p)}{\partial(v_4p_4)}\right)(0) \qquad \Sigma_3 \;=\; \left(\frac{\partial\Sigma(p)}{\partial(v_3p_3)}\right)(0) \qquad (84)$$

The analytical expressions for Σ_4 and Σ_3 are readily computable from eq.(82).

If we impose on-shell renormalization conditions along with the normalization of the velocity $v^2 = 1$, it is easy to check that up to order α_s [16]

$$
\begin{aligned}
\delta M &= -\Sigma_0 && (85)\\
Z_Q &= 1 + v_4^2\,\Sigma_4 - v_3^2\,\Sigma_3 \\
&= 1 + \left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\left[-4\,\log(a\lambda) + v_4^2\,\Sigma_{40}^{(0)} + v_4^2\sum_{i=0}^{\infty}\left(\frac{v_3}{v_4}\right)^{2i}\left\{\Sigma_{40}^{(2i+2)} - \Sigma_{30}^{(2i+1)}\right\}\right] && (86)\\
Z_v^{-1} &= 1 + v_4^2\,\Sigma_4 - v_4^2\,\Sigma_3 \\
&= 1 + \left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\,v_4^2\sum_{i=0}^{\infty}\left(\frac{v_3}{v_4}\right)^{2i}\left\{\Sigma_{40}^{(2i)} - \Sigma_{30}^{(2i+1)}\right\} && (87)
\end{aligned}
$$

where Z_v is the renormalization of the heavy quark velocity, first introduced in ref.[16], defined by

$$
v_3^R = Z_v^{-1}\,v_3^B \qquad (88)
$$

As we will see in the next section, Z_v is a lattice effect that originates from the fact that the wave function and mass renormalizations are not sufficient to match the lattice and continuum amplitudes. In this sense, this 'velocity' renormalization can be interpreted also as the matching constant necessary to reproduce the physical amplitudes in the continuum from the ones on the lattice. We will return to this subject in the next section.

For future use, it is convenient to write Z_Q and Z_v as

$$
\begin{aligned}
Z_Q &= 1 + \left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\left[-4\,\log(a\lambda) + v_4^2\sum_{i=0}^{\infty}\left(\frac{v_3}{v_4}\right)^{2i} Z_Q^{(2i)}\right] && (89)\\
Z_v &= 1 + \left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\,v_4^2\sum_{i=0}^{\infty}\left(\frac{v_3}{v_4}\right)^{2i} Z_v^{(2i)} && (90)
\end{aligned}
$$

and the reader can find the numerical values of $Z_{Q,v}^{(m)}$ in Table A.2.

To finish this section we wish to briefly comment on some particularly interesting characteristics of Z_Q and Z_v. We observe that Z_Q is logarithmically divergent so that the corresponding anomalous dimension coincides with the one in the continuum and is independent of the velocity. On the other hand, Z_v is finite because the logarithmically divergent terms coming from the self-energy cancel out exactly in eq.(87) order by order in the velocity. Finally, since lattice regularization breaks the $O(4)$ symmetry, the renormalization constants of the effective theory depend on the velocity of the heavy quark. This does not happen in a Lorentz invariant theory because there all quantities must depend on Lorentz invariants.

5.4 Renormalization and matching of $(\vec{v} \cdot \vec{D})$

The 'velocity' operator $(\vec{v} \cdot \vec{D})$ does not renormalize multiplicatively and, in general, we need to perform subtractions of terms that diverge as powers of the ultra-violet cut-off a^{-1}. Specifically, we will demonstrate that $(\vec{v} \cdot \vec{D})$ mixes under renormalization with the operator $\mathbf{K}(x) = Q^\dagger(x) D_4(x) Q(x)$ through a coefficient free of power divergences and with the operator $\mathbf{1}(x) = Q^\dagger(x) Q(x)$ whose coefficient diverges as $1/a$.

It is convenient to proceed order by order in the velocity because in this way the matching can be understood better.

Consider a single insertion of $(\vec{v} \cdot \vec{D})$. The vanishing of the corresponding amplitude at zero external momentum (eqs.(42) and (47)) imply that $(\vec{v} \cdot \vec{D})$ does not mix with the operator $\mathbf{1}(x)$ with a linearly divergent coefficient (i.e. proportional to $1/a$). There is only a multiplicative renormalization of $(\vec{v} \cdot \vec{D})$. In order to obtain it, we need the wave-function renormalization constant Z of the heavy quark in the static theory. It has been computed by many authors [26]

$$Z \;=\; 1 - i \left(\frac{\partial \Sigma}{\partial p_4} \right)_0 \;=\; 1 + \left(\frac{\alpha_s}{\pi} \right) \frac{C_F}{4} \left[-2 \log(a\lambda)^2 + \Sigma_{40}^{(0)} \right] \tag{91}$$

where $\Sigma_{40}^{(0)} = 24.48$ (see eq.(82)). Then, the one-loop matrix element of the bare operator $(\vec{v} \cdot \vec{D})$ between heavy quark states is given by

$$\begin{aligned} \langle (\vec{v} \cdot \vec{D}) \rangle \;&=\; -\left(1 - \frac{1}{v_3} \left(\frac{\partial A_3}{\partial p_3} \right)_0 - \frac{1}{v_3} \left(\frac{\partial A_4}{\partial p_3} \right)_0 - i \left(\frac{\partial \Sigma}{\partial p_4} \right)_0 \right) (v_3 p_3) \;+\; \ldots \\ &=\; -\left(1 + \left(\frac{\alpha_s}{\pi} \right) \frac{C_F}{4} \left[\Sigma_{40}^{(0)} - \Sigma_{30}^{(1)} \right] \right) (v_3 p_3) \;+\; \ldots \end{aligned} \tag{92}$$

The dots indicate terms which vanish as p^2 for $p \to 0$, and therefore do not contribute to the on-shell renormalization. As anticipated in the previous section, the ultraviolet divergence of the vertex correction cancels the one of the field renormalization constant Z, leaving a finite term. This occurs because the operator $(\vec{v} \cdot \vec{D})$ is conserved in the static theory.

This result can easily be generalized for an odd number $2m+1$ of insertions of the operator $(\vec{v} \cdot \vec{D})$. In fact, from (76) we learn that it does not mix with the operator $\mathbf{1}$ because the corresponding amplitude vanishes at $p = 0$ due to the spatial parity invariance of the theory. Therefore, two-quark matrix elements of $(\vec{v} \cdot \vec{D})$ do not contain linearly divergent terms proportional to $\mathbf{1}$. The same reasoning applies to the mixing with the operator $\mathbf{K}$. Thus, $(\vec{v} \cdot \vec{D})$ renormalizes multiplicatively with a finite renormalization constant at any odd order in the velocity.

Let us consider now the renormalization of the double insertion of $(\vec{v}\cdot\vec{D})$. The amplitudes at zero external momentum are now non-vanishing,

$$B_3(0)\ +\ B_4(0)\ \neq\ 0, \tag{93}$$

implying that there is a mixing of the double insertion of $(\vec{v}\cdot\vec{D})$ with the operator $\mathbf{1}(x)$ with a linearly divergent coefficient. There is also a mixing of the double insertion of $(\vec{v}\cdot\vec{D})$ with the operator $\mathbf{K}(x)$, because

$$\left(\frac{\partial B_3}{\partial p_4}\right)_0+\left(\frac{\partial B_4}{\partial p_4}\right)_0\ \neq\ 0 \tag{94}$$

The mixing is finite because the logarithmic divergences of $(\partial B_3/\partial p_4)_0$ and $(\partial B_4/\partial p_4)_0$ cancel each other. This is true at any order in the velocity, as we demonstrated before.

We have therefore the one-loop result

$$\langle Q^\dagger(\vec{v}\cdot\vec{D})Q\,Q^\dagger(\vec{v}\cdot\vec{D})Q\rangle\ =\ \frac{1}{a}\left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\ v_3^2\ \Sigma_0^{(2)}\,\langle\mathbf{1}\rangle\ +\ \left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\ v_3^2\ \Sigma_{40}^{(2)}\ \langle\mathbf{K}\rangle+\cdots \tag{95}$$

where the dots indicate terms which do not contribute to the on-shell renormalization.

As before, these results can be extended to any order in the velocity by means of eqs.(76) and (78). We only give the final result for the one-loop renormalized lattice operator $(\vec{v}\cdot\vec{D})$

$$\begin{aligned}
[(\vec{v}\cdot\vec{D})]^{(1)}_{Latt}\ \ &=\ \ \left(1+\left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\,v_4^2\,v_3\left[\sum_{i=0}^{\infty}\left(\frac{v_3}{v_4}\right)^{2i}\left\{\Sigma_{30}^{(2i+1)}\ -\ \Sigma_{40}^{(2i)}\right\}\right]\right)[(\vec{v}\cdot\vec{D})]^{(0)}\\
&+\ \ \frac{1}{a}\left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\,v_4\left[\sum_{i=0}^{\infty}\left(\frac{v_3}{v_4}\right)^{2i}\ \Sigma_0^{(2i)}\right][\mathbf{1}]^{(0)}\\
&+\ \ \left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\left[\sum_{i=0}^{\infty}\left(\frac{v_3}{v_4}\right)^{2i}\ \Sigma_{40}^{(2i)}\right][\mathbf{K}]^{(0)}
\end{aligned} \tag{96}$$

where the superscripts (0) denote bare operators.

To proceed further, we match the amplitudes of the bare lattice SHQET onto the amplitudes of the $\overline{MS}$-renormalized HQET (i.e. non expanded in v_3). To do this we need to know the two-quark amplitude of the operator $(\vec{v}\cdot\vec{D})$ for an external momentum configuration on-shell. By direct computation, we observe that order by order in the velocity v_3 the sum of the relevant loop diagrams with insertions of v_3-vertices vanish. Of course, there is a physical reason for this to happen: the operator $(\vec{v}\cdot\vec{D})$ is conserved in the continuum static theory, therefore it does not get renormalized by interactions with gluons. Thus, we can write

$$[(\vec{v}\cdot\vec{D})]^{(1)}_{\overline{MS}}\ =\ [(\vec{v}\cdot\vec{D})]^{(0)}\ +\ O(\alpha_s^2) \tag{97}$$

In other words, the one-loop wave function renormalization of a heavy quark is independent of its velocity due to the fact that dimensional regularization is a covariant regularization.

We can now perform the continuum-lattice matching by computing the ratio of the continuum amplitude to the lattice one. From eq.(96) and (97) we have that the physical operator $(\vec{v}\cdot\vec{D})$ is related to the lattice bare one by

$$
\begin{aligned}
[(\vec{v}\cdot\vec{D})]_{\overline{MS}} &= \left(1+\left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}v_4^2\left[\sum_{i=0}^{\infty}\left(\frac{v_3}{v_4}\right)^{2i}\left\{\Sigma_{30}^{(2i+1)} - \Sigma_{40}^{(2i)}\right\}\right]\right)[(\vec{v}\cdot\vec{D})]_{Latt} \\
&- \frac{1}{a}\left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}v_4\left[\sum_{i=0}^{\infty}\left(\frac{v_3}{v_4}\right)^{2i}\Sigma_0^{(2i)}\right][\mathbf{1}]_{Latt} \\
&- \left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\left[\sum_{i=0}^{\infty}\left(\frac{v_3}{v_4}\right)^{2i}\Sigma_{40}^{(2i)}\right][\mathbf{K}]_{Latt} \\
&\equiv Z_v\left[[(\vec{v}\cdot\vec{D})]_{Latt} - \frac{c_1}{a}[\mathbf{1}]_{Latt} - c_2[\mathbf{K}]_{Latt}\right]
\end{aligned}
\tag{98}
$$

with obvious notation. As anticipated, we learn from the previous equation that Z_v can be interpreted as the lattice-continuum matching constant of the operator $(\vec{v}\cdot\vec{D})$.

An equivalent way of performing the matching is to expand the $\overline{MS}$-renormalized propagator in the HQET and compare it order by order in the v_3 with the propagator in the SHEQT on the lattice. In the continuum we have

$$
\begin{aligned}
iS(k) &= \frac{Z_{\overline{MS}}}{iv_4k_4+v_3k_3} = \frac{Z_{\overline{MS}}}{ik_4+\epsilon} + \frac{\sqrt{Z_{\overline{MS}}}}{ik_4+\epsilon}(-v_3k_3)\frac{\sqrt{Z_{\overline{MS}}}}{ik_4+\epsilon} \\
&+ \frac{\sqrt{Z_{\overline{MS}}}}{ik_4+\epsilon}(-v_3k_3)\frac{1}{ik_4+\epsilon}(-v_3k_3)\frac{\sqrt{Z_{\overline{MS}}}}{ik_4+\epsilon} + \frac{\sqrt{Z_{\overline{MS}}}}{ik_4+\epsilon}\frac{-iv_3^2}{2}k_4\frac{\sqrt{Z_{\overline{MS}}}}{ik_4+\epsilon} + \cdots
\end{aligned}
\tag{99}
$$

where $Z_{\overline{MS}}$ is the heavy quark field renormalization constant

$$
Z_{\overline{MS}} = 1 + \left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\,2\log(\mu/\lambda)^2
\tag{100}
$$

The bare lattice propagator in the SHQET is instead given (near the mass-shell) by:

$$
\begin{aligned}
i\tilde{S}(k) &= \frac{Z}{ik_4+\epsilon} + \frac{\sqrt{Z}}{ik_4+\epsilon}(-v_3k_3)\left(1+\left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}[\Sigma_{40}^{(0)} - \Sigma_{30}^{(1)}]\right)\frac{\sqrt{Z}}{ik_4+\epsilon} \\
&+ \frac{\sqrt{Z}}{ik_4+\epsilon}(-v_3k_3)\left(1+\left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}[\Sigma_{40}^{(0)} - \Sigma_{30}^{(1)}]\right)\frac{1}{ik_4+\epsilon}(-v_3k_3) \\
&\times \left(1+\left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}\left\{[\Sigma_{40}^{(0)} - \Sigma_{30}^{(1)}] + \frac{1}{a}v_3^2\,\Sigma_0^{(2)} + v_3^2\,\Sigma_{40}^{(2)}\,ik_4\right\}\right)\frac{\sqrt{Z}}{ik_4+\epsilon} \\
&+ \frac{\sqrt{Z_{\overline{MS}}}}{ik_4+\epsilon}\frac{-iv_3^2}{2}k_4\frac{\sqrt{Z_{\overline{MS}}}}{ik_4+\epsilon} + \cdots
\end{aligned}
\tag{101}
$$

where Z is the field renormalization constant of the static lattice theory given in eq.(91) (we omit for simplicity the mass renormalization).

Matching at lowest order in v_3 (static approximation) is realized by introducing a matching constant ζ of the bare lattice regulated field onto the $\overline{MS}$ renormalized field

$$Q_{\overline{MS}} \;=\; \zeta\; Q_L \tag{102}$$

where

$$\zeta \;=\; \frac{Z_{\overline{MS}}}{Z} \;=\; 1 + \left(\frac{\alpha_s}{\pi}\right)\, \frac{C_F}{4}\, [2\log(a\mu)^2 - \Sigma_{40}^{(0)}] \tag{103}$$

At order v_3, we must introduce a matching constant Z_v defined by

$$(\vec{v}\cdot\vec{D})_{\overline{MS}} \;=\; Z_v\; (\vec{v}\cdot\vec{D})_{BL} \tag{104}$$

The comparison of eqs.(99) and (101) gives

$$Z_v \;=\; 1 + \left(\frac{\alpha_s}{\pi}\right)\, \frac{C_F}{4}\, [\Sigma_{30}^{(1)} - \Sigma_{40}^{(0)}] \tag{105}$$

Matching at order v_3^2 requires to subtract from the double insertion of $(\vec{v}\cdot\vec{D})$, the contribution proportional to $\mathbf{K}$ and the one proportional to $\mathbf{1}$, since they are absent in the HQET propagator. This means that there is a mixing of these operators in the lattice-continuum matching. Performing the subtraction above, we reproduce (95).

This procedure can be iterated to higher orders in the velocity v_3 leading to eq.(97).

6 Renormalization of the heavy quark current

In this section, we deal with the renormalization of the heavy-quark current

$$J(x) \;=\; Q^{\dagger}(x,v)\,\Gamma\, Q(x,v') \tag{106}$$

describing the transition of a heavy quark with velocity v into a heavy quark of velocity v'. Γ stands for any of the 16 Dirac matrices. We specialize our computation to the most interesting case $v' = (1,\vec{0})$ and $v = (0,0,v_3,\sqrt{1+v_3^2})$.

We will demonstrate that the weak current $J(x)$ renormalizes multiplicatively with a coefficient that is only logarithmically divergent. We will show explicitly that the one-loop anomalous dimension of $J(x)$ depends on the velocity and coincides order by order in the velocity with the one computed within HQET in the continuum.

For the sake of clarity, we divide this section in two parts. In the first one, we compute the on-shell lattice matrix element of $J(x)$ between heavy quark states up to order $O(v_3^2)$. The second subsection is devoted to extend the previous result to all orders in the velocity. It is there where we will re-obtain the velocity-dependent one-loop anomalous dimension by summing all the diagrams with insertions of the operator $(\vec{v}\cdot\vec{D})$.

6.1 Matrix element of the current up to $O(v_3^2)$

We start by considering the renormalization of the weak current $J(x)$ with one insertion of the operator $(\vec{v}\cdot\vec{D})$. We study the Green's function

$$G(x,z) \;=\; \int \mathrm{d}^4\mathrm{y}\mathrm{d}^4\mathrm{w}\,\langle 0 \mid \mathrm{T}\;[\mathrm{Q(x)\,Q(y)}(\vec{\mathrm{v}}\cdot\vec{\mathrm{D}})\mathrm{Q}^\dagger\mathrm{(y)\,J(w)\,Q}^\dagger\mathrm{(z)}] \mid 0\rangle \tag{107}$$

The only non-vanishing diagram involved is drawn in Fig.5. The amplitude of diagram D.1 is given by

$$D_1(p,q) \;=\; \frac{g^2 C_F v_3}{2}\; e^{iq_4-2ip_4}\int \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4}\frac{e^{-ik_4}\sin(k_3+p_3)}{1-e^{-i(k_4+p_4)}+i\epsilon}\frac{1}{\Delta_1(k)} \tag{108}$$

where p is the final momentum of Q and q is the (incoming) momentum of J. There is a potential logarithmic divergence, which is absent because the integral vanishes by parity at zero external momenta

$$D_1(p=0,q=0) \;=\; 0 \tag{109}$$

Therefore, there is not any additional renormalization of the Green's functions of the form (107).

Consider now the renormalization of the Green functions of $J(x)$ containing a double insertion of $(\vec{v}\cdot\vec{D})$. The relevant diagram is given in Fig.5. The amplitude for D.2 is given, at zero external momenta, by

$$D_2 \;=\; -\frac{g^2 C_F v_3^2}{6}\;\int \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4}\frac{\eta(\vec{k})\;e^{-ik_4}}{(1-e^{-ik_4}+\epsilon)^4}\frac{1}{\Delta_1(k)} \tag{110}$$

With the technique described in sec. 5.1, the previous integral is transformed into

$$\begin{aligned} D_2 \;=\;& \frac{g^2 C_F}{16\pi^2}v_3^2\Bigg(-\frac{1}{36\pi^2}\int d^4k\frac{\eta(\vec{k})}{\Delta_1(k)^3}[\;\cos k_4+2\cos 2k_4+\cos 3k_4\;] \\ &-\frac{1}{36\pi^2}\int d^4k\frac{\eta(\vec{k})}{\Delta_1(k)^2}[\;3/2+3\cos k_4+2\cos 2k_4\;]-\frac{1}{12\pi}\int \mathrm{d}^3\mathrm{k}\frac{\eta(\vec{\mathrm{k}})}{\Delta_1(0,\vec{\mathrm{k}})^2}\Bigg) \end{aligned} \tag{111}$$

The logarithmic singularity of the amplitude is entirely contained in the first integral. The numerical computation gives

$$D_2 \;=\; \left(\frac{\alpha_s}{\pi}\right)\;\frac{C_F}{4}\;\;v_3^2\,[\;\frac{2}{3}\log(a\lambda)^2+d_2\;] \tag{112}$$

where $d_2 = -5.022$.

Adding to (112) the contribution from the external wave-function renormalization Z_Q, we obtain that the matrix element of the current $J(x)$ between on-shell heavy-quark states is

$$\begin{aligned}\langle c, v_3 \mid J \mid b, \vec{0} \rangle &= 1 - \left(\frac{\alpha_s}{\pi}\right) \frac{C_F}{4} \left[Z_\xi^{(0)} + v_3^2 \, Z_\xi^{(2)} - \frac{4}{3} v_3^2 \log(a\lambda) \right] \\ &+ O(\alpha_s \, a \, , v_3^3) \end{aligned} \tag{113}$$

where $Z_\xi^{(0)} = -19.95$ is the old result for a static heavy quark and $Z_\xi^{(2)} = -4.653$. The reason for the introduction of the constants $Z_\xi^{(n)}$ will be apparent in section 6.3.

Note that the logarithmic divergence from the vertex diagram for two static heavy quarks (lower order in v_3) exactly cancels the one from the external wave-function renormalization resulting in a finite lowest order correction to the matrix element of the current $Z_\xi^{(0)}$. The physical reason for this to happen is that the flavour conserving current, i.e. the current $J(x)$ for equal velocities $v = v'$ or equivalently $v_3 = 0$, is conserved in the HQET and so its anomalous dimension must be zero. Therefore, the anomalous dimension of the current $J(x)$ starts from v_3^2 in an expansion in the velocity.

6.2 Matrix element of the current beyond $O(v_3^2)$

The only non-vanishing diagram we need to calculate now has the same structure as those in Fig.5 but with m insertions of the operator $(\vec{v} \cdot \vec{D})$. We will denote it by E. All other possible one-particle irreducible diagrams vanish due to parity. Again, we do not consider the insertion of v_4-vertices because the net effect of all such vertices is to multiply the original diagram by $1/v_4^m$, with m the number of v_3-vertices. The demonstration of this assertion is similar to the case of the self-energy and so we do not repeat it here.

The amplitude corresponding to the diagram E is

$$\begin{aligned} E(m,p) &= -\left(\frac{\alpha_s}{\pi}\right) C_F \, (-v_3)^m \, e^{-2ip_4} \, \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d^3k}{(2\pi)^3} \sin^m(p_3 - k_3) \\ &\times \oint \frac{dz}{2\pi i \, z} \, \frac{z}{[1 + \epsilon - e^{-ip_4} \, z]^{m+2}} \, \frac{-z}{(z - z_-)(z - z_+)} \end{aligned} \tag{114}$$

where $z = e^{ik_4}$ and the contour integral is along the unit circle.

The non-vanishing term as a goes to zero is $E(p = 0)$, which contains a linear divergence. First derivatives with respect to the external momentum give rise to terms of order $O(a)$ that do not contribute to the on-shell renormalization. The computation reduces to the evaluation of the contour integral over z which can be easily performed taking into account that only the pole $z = z_-$ lies in the unit circle. Furthermore, the ϵ-prescription tells us that the pole of the quark propagator does not contribute to the contour integral when the Cauchy's theorem is used. The result is,

$$\begin{aligned} E(2m+1, p=0) &= 0 \quad \text{by parity} \\ E(2m, p=0) &= -\left(\frac{\alpha_s}{\pi}\right) C_F\, v_3^{2m} \frac{2}{4^{m+1}} \\ &\times \left[\mathrm{Si}^{(20)}(m-1) + \mathrm{Si}^{(11)}(m-1) + \mathrm{Si}^{(21)}(m-1) \right] \end{aligned} \tag{115}$$

For $m = 0$, the static case, the amplitude greatly simplifies

$$E(0, p=0) = -\left(\frac{\alpha_s}{\pi}\right) \frac{C_F}{4} \mathrm{Id}^{(11)}(0) \tag{116}$$

The integrals $\mathrm{Si}^{(\alpha\beta)}(m)$ and $\mathrm{Id}^{(11)}(0)$ are defined and their numerical values tabulated in appendix A.

The matrix element of the current $J(x)$ between heavy-quark states is logarithmically divergent for $m = 0$ and also for $m > 0$ because so is $\mathrm{Si}^{(20)}(m-1)$. The coefficients of the logarithms can easily be extracted order by order in the velocity from eqs.(B.10) and (B.12). Adding the contribution from the wave-function renormalization of the external states (see eq.(86) and (89)), we have that the matrix element of the current can be written as

$$\begin{aligned} \langle c, v_3 \mid J \mid b, \vec{0}\ \rangle = 1 &+ \left(\frac{\alpha_s}{\pi}\right) \frac{C_F}{4} \sum_{i=0}^{\infty} \Big\{ -v_3^{2i}\, Z_\xi^{(2i)} \\ &+ \left(\frac{v_3}{v_4}\right)^{2i} 2 \left[\frac{1}{(2i+1)} - \delta_{i,0} \right] \log(a\lambda)^2 \Big\} + O(\alpha_s\, a) \end{aligned} \tag{117}$$

where $Z_\xi^{(m)}$ can easily be evaluated from eqs.(115), (116) and (89). In order to simplify the computation in section 6.4 of the relation between the Isgur-Wise function on the lattice and in the continuum $\overline{MS}$, we have expanded the finite contributions in powers of v_3 instead of v_3/v_4. This is achieved by noting that

$$\left(\frac{v_3}{v_4}\right)^{2m} = \sum_{j=0}^{\infty} (-)^j \frac{(i+j-1)!}{j!\,(i-1)!} v_3^{2(i+j)} \tag{118}$$

For the numerical values of constants $Z_\xi^{(n)}$ we refer the reader to Table A.2.

The only subtlety in this calculation is the fact that the wave function renormalization at lowest order in the velocity (i.e. the static case) contributes to the matrix element with a coefficient twice the one of higher velocity orders, which is 1/2. The reason is that we consider the b quark static and the c quark moving with a small velocity v_3. Therefore, only the latter, as a consequence of its interaction with the gluon field, gets a velocity dependent wave function renormalization which lowest order is the corresponding to a static heavy quark.

The interesting thing is that the infinite sum in front of $\log(a\lambda)$ can be evaluated simply recalling that

$$\frac{1}{2u} \log\left(\frac{1+u}{1-u}\right) = \sum_{i=0}^{\infty} \frac{u^{2i}}{(2i+1)} \tag{119}$$

Substituting (119) into (117), we get

$$\begin{aligned} \langle c, v_3 \mid J \mid b, \vec{0}\, \rangle \;=\; 1 \;&+\; \left(\frac{\alpha_s}{\pi}\right) \frac{C_F}{4} \left[\left(-2 + \frac{v_4}{v_3} \log\left(\frac{v_4+v_3}{v_4-v_3}\right)\right) \log(a\lambda)^2 \;-\; \sum_{i=0}^{\infty} \left\{ v_3^{2i}\, Z_\xi^{(2i)} \right\} \right] \\ &+\; O(\alpha_s\, a) \end{aligned} \tag{120}$$

It should be stressed that the infrared structure of the matrix element of the heavy-heavy current is the same as the one evaluated within the continuum Georgi's theory in ref.[27], as it should be. In other words, the expansion in the velocity reproduces the correct infrared behaviour of the theory once we sum all orders in the velocity. This is a check of the consistency of our approach.

Another check is provided by the comparison of the numerical value of the constant $Z_\xi^{(2)}$ listed in Table A.2 (computed by direct contour integration) and the one given below eq.(113) (computed by integration by parts). They coincide within error bars.

6.3 Lattice-continuum matching

In order to match the two-quark amplitude of the current to its counterpart in the continuum, we need to compute the matrix element of the current in the HQET renormalized in the $\overline{MS}$ scheme. This has already been done by the authors of ref.[27] and here we only quote their final result

$$\begin{aligned} \langle c, v_3 \mid J \mid b, \vec{0}\, \rangle_{\overline{MS}} \;=\; 1 \;&+\; \left(\frac{\alpha_s}{\pi}\right) \frac{C_F}{4} \left(2 - \frac{v_4}{v_3} \log\left(\frac{v_4+v_3}{v_4-v_3}\right)\right) \log(\mu/\lambda)^2 \\ &+\; O(\alpha_s^2) \end{aligned} \tag{121}$$

where μ is the renormalization point.

By forming the ratio of (121) to (120) we get the factor that relates the matrix elements of the heavy-quark current $J(x)$ in the lattice and in the continuum $\overline{MS}$ renormalization schemes

$$\begin{aligned} \frac{\langle c, v_3 \mid J \mid b, \vec{0}\, \rangle_{\overline{MS}}}{\langle c, v_3 \mid J \mid b, \vec{0}\, \rangle_{Latt}} = 1 \;&+\; \left(\frac{\alpha_s}{\pi}\right) \frac{C_F}{4} \left[\left(-2 + \frac{v_4}{v_3} \log\left(\frac{v_4+v_3}{v_4-v_3}\right)\right) \log(a\mu)^2 \right. \\ &+\; \left. \sum_{i=0}^{\infty} \left\{ v_3^{2i}\, Z_\xi^{(2i)} \right\} \right] + O(\alpha_s^2) \end{aligned} \tag{122}$$

As expected, the infrared regulator λ disappears in the matching because the lattice theory and its counterpart in the continuum have the same infrared behaviour. It is the ultraviolet one that is different and (122) takes this discrepancy into account.

Let us discuss now the on-shell renormalization of the lattice SHQET in the real space [28] instead of momentum space [26] as we have done up to now. These renormalization schemes differ on the lattice and the relation between them has been clarified in [29]. A clear exposition can be found in [16] which we will follow almost verbatim here.

Consider the bare propagator of the heavy quark moving on the lattice with a velocity v_3 along the z-axis as a function of time and momentum $\vec{p}$. We will call it $iH(t,\vec{p})$. This propagator can be obtained by performing the Fourier transform with respect to p_4, the fourth component of the external momentum, of the propagator in the momentum space (83). For large euclidean time and in the continuum limit, $iH(t,\vec{p})$ reduces to

$$iH(t,\vec{p}) \,=\, Z_Q \,\frac{\theta(t)}{v_4^R}\, \exp[-(t+1)/v_4^R\; (\delta M \,+\, v_3^R \cdot p_3)\,] \tag{123}$$

Note that for the momentum space renormalization conditions (85) to (87), Z_Q is multiplied by an exponential with $(t+1)$ instead of t in the heavy quark propagator.

On the other hand, in lattice simulations one fits the correlation functions to an exponential evolution in euclidean time with t instead of $(t+1)$ (real space renormalization scheme). Therefore, if we do not modify the momentum space renormalization conditions appropriately, we will not take into account the correct wave function renormalization giving rise to a wrong lattice-continuum matching. The solution is to take a shifted wave function renormalization $\overline{Z}_Q$ related to the old one by [26]

$$\overline{Z}_Q \,=\, Z_Q \,-\, \frac{\delta M}{v_4} \tag{124}$$

which tells us that he discrepancy between the momentum and the real space renormalization schemes is finite and is given by the mass renormalization. In Table A.2, we have tabulated the values of the renormalization constants $\overline{Z}_Q$ and $\overline{Z}_\xi$ in the real space lattice scheme.

7 The Isgur-Wise function

In this section we determine the relation between the value of the derivatives of the Isgur-Wise function at the zero recoil point, $\xi^{(n)}(1)$, measured on the lattice and its physical counterparts in the continuum $\overline{MS}$.

Isgur-Wise derivatives	Numerical Coefficients			
	1	$\xi^{(1)}_{Lat}(1)$	$\xi^{(2)}_{Lat}(1)$	$\xi^{(3)}_{Lat}(1)$
$\triangle\xi^{(1)}(a^{-1})$	-9.31	-19.95	0.0	0.0
$\triangle\xi^{(2)}(a^{-1})$	3.44	-18.62	-19.95	0.0
$\triangle\xi^{(3)}(a^{-1})$	88.20	10.31	-27.92	-19.95
$\triangle\overline{\xi}^{(1)}(a^{-1})$	-24.88	0.0	0.0	0.0
$\triangle\overline{\xi}^{(2)}(a^{-1})$	17.35	-49.76	0.0	0.0
$\triangle\overline{\xi}^{(3)}(a^{-1})$	10.56	52.04	-74.64	0.0

Table 1: Numerical values of the constants determining the continuum–lattice matching of the first derivatives of the Isgur-Wise function. A factor $\left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}$ multiplying all entries is understood.

In fact, taking $\mu = a^{-1}$ in (122), we find

$$\frac{\xi_{\overline{MS}}(v_4)}{\xi_{Lat}(v_4)} = 1 + \left(\frac{\alpha_s(a^{-1})}{\pi}\right)\frac{C_F}{4}\left\{Z^{(0)}_{\xi} + Z^{(2)}_{\xi}\,v_3^2 + Z^{(4)}_{\xi}\,v_3^4 + \cdots\right\} \tag{125}$$

where the matching constants $Z^{(n)}_{\xi}$ are defined in eq.(117). Note that by setting $\mu = a^{-1}$ we have got ride of the logarithms that appear in the lattice-continuum matching.

Substituting the expansions of both the Isgur-Wise function in the continuum and the one on the lattice in powers of v_3 into (125) and demanding consistency order by order in v_3, we get

$$\triangle\xi^{(1)}(a^{-1}) = \left(\frac{\alpha_s(a^{-1})}{\pi}\right)\frac{C_F}{4}\left\{Z^{(0)}_{\xi}\,\xi^{(1)}_{Lat}(1) + 2\,Z^{(2)}_{\xi}\right\} \tag{126}$$

$$\begin{aligned}\triangle\xi^{(2)}(a^{-1}) &= \left(\frac{\alpha_s(a^{-1})}{\pi}\right)\frac{C_F}{4}\left\{Z^{(0)}_{\xi}\,\xi^{(2)}_{Lat}(1) + 4\,Z^{(2)}_{\xi}\,\xi^{(1)}_{Lat}(1)\right.\\ &+ \left.2\,Z^{(2)}_{\xi} + 8\,Z^{(4)}_{\xi}\right\}\end{aligned} \tag{127}$$

$$\begin{aligned}\triangle\xi^{(3)}(a^{-1}) &= \left(\frac{\alpha_s(a^{-1})}{\pi}\right)\frac{C_F}{4}\left\{Z^{(0)}_{\xi}\,\xi^{(3)}_{Lat}(1) + (6\,Z^{(2)}_{\xi} + 24\,Z^{(4)}_{\xi})\,\xi^{(1)}_{Lat}(1)\right.\\ &+ \left.6\,Z^{(2)}_{\xi}\,\xi^{(2)}_{Lat}(1) + 24\,Z^{(4)}_{\xi} + 48\,Z^{(6)}_{\xi}\right\}\end{aligned} \tag{128}$$

with $\xi^{(n)}(1)$ being the nth derivative of the Isgur-Wise function with respect to v_4 at the zero recoil point $v_4 = 1$ and $\triangle\xi^{(n)}(\mu_0) = \xi^{(n)}_{\overline{MS}}(1)\mid_{\mu=\mu_0} - \xi^{(n)}_{Lat}(1)$.

Eqs.(126) to (128) are our most important results. They give the one-loop relation at the scale $\mu = a^{-1}$ between the lattice measures of the derivatives of the Isgur-Wise function and their physical values. It should be stressed that equivalent expressions can be obtain for the real space

Isgur-Wise derivatives	R. G. correction		
	$a^{-1} = 2$ GeV	$a^{-1} = 4$ GeV	$a^{-1} = 6$ GeV
$\triangle Z_\xi^{(1)}(\overline{m})$	0.668	2.111	3.028
$\triangle Z_\xi^{(2)}(\overline{m})$	−0.248	−0.789	−1.135
$\triangle Z_\xi^{(3)}(\overline{m})$	0.155	0.489	0.701

Table 2: Renormalization group (R. G.) corrections to the constants determining the continuum–lattice matching of the first derivatives of the Isgur-Wise function at the scale $\mu = \overline{m}$ for several lattice spacings a. The two-loop anomalous dimension of the heavy-heavy quark current has been properly included. A factor $\left(\frac{\alpha_s}{\pi}\right) \frac{C_F}{4}$ multiplying all entries is understood.

renormalization scheme by replacing $Z_\xi^{(n)}$ by $\overline{Z}_\xi^{(n)}$. In Table 1, we give the numerical values of the coefficients of $\xi_{Lat}^{(n)}(1)$ in $\triangle\xi^{(n)}(a^{-1})$ both for the momentum and real space (denoted with a bar) renormalization schemes.

As we mention in Sec.4, the values of the matching coefficients in Table 1 depend on the continuum renormalization scheme. Although we expect their numerical values not to change very much in a different renormalization scheme, consistency requires to properly include the contribution of the two-loop anomalous dimension. In addition, we give the renormalization group evolution of the derivatives of the Isgur-Wise function from the scale a^{-1} to a generic renormalization point μ.

The one-loop anomalous dimension γ_1 of the heavy-heavy quark current is velocity dependent and has been calculated in eq.(120). On the other hand, the two-loop anomalous dimension γ_2 has been computed in ref.[30] in the $\overline{MS}$ scheme. Expanding both γ_1 and γ_2 as a power series in v_3^2 and inserting the result in eq.(35), we find that the renormalization group corrections $\triangle Z_\xi^{(n)}(\mu)$ to the matching constants $Z_\xi^{(n)}$ are

$$\begin{aligned}
\left(\frac{\alpha_s(a^{-1})}{\pi}\right) \frac{C_F}{4} \triangle Z_\xi^{(2)}(\mu) &= -\frac{C_F}{3}\frac{1}{\beta_1}\Big\{ \log(\alpha_s(\mu)/\alpha_s(a^{-1})) \\
&+ \left(\frac{\alpha_s(a^{-1})}{\pi} - \frac{\alpha_s(\mu)}{\pi}\right)\left[2\pi^2 + \frac{5}{18}N_F - \frac{29}{6} + \frac{\beta_2}{\beta_1}\right]\Big\} \qquad (129)\\
\left(\frac{\alpha_s(a^{-1})}{\pi}\right) \frac{C_F}{4} \triangle Z_\xi^{(4)}(\mu) &= -\frac{C_F}{3}\frac{1}{\beta_1}\Big\{ -\frac{2}{5}\log(\alpha_s(\mu)/\alpha_s(a^{-1})) \\
&+ \left(\frac{\alpha_s(a^{-1})}{\pi} - \frac{\alpha_s(\mu)}{\pi}\right)\left[-\frac{4\pi^2}{5} - \frac{1}{9}N_F + \frac{61}{40} - \frac{2}{5}\frac{\beta_2}{\beta_1}\right]\Big\} \qquad (130)
\end{aligned}$$

$$\left(\frac{\alpha_s(a^{-1})}{\pi}\right)\frac{C_F}{4}\triangle Z_\xi^{(6)}(\mu) = -\frac{C_F}{3}\frac{1}{\beta_1}\left\{\frac{8}{35}\log(\alpha_s(\mu)/\alpha_s(a^{-1})) + \left(\frac{\alpha_s(a^{-1})}{\pi} - \frac{\alpha_s(\mu)}{\pi}\right)\left[\frac{\pi^2}{35} + \frac{4}{63}N_F - \frac{43501}{37800} + \frac{8}{35}\frac{\beta_2}{\beta_1}\right]\right\} \quad (131)$$

In Table 2, we report the numerical renormalization group corrections to the constants $Z_\xi^{(n)}$ for several values of a^{-1} at the physical meaningful scale $\mu = \overline{m}$ where $\overline{m} = m_B m_D/(m_B + m_D) \approx 1.4$ GeV, i.e. the reduced mass of the B- and D-meson. The number of active quarks is 3 because both the b and the c quarks are taken to be static sources of color. The values in Table 2 must be added to those of Table A.1 to obtain the matching constants at the scale $\overline{m}$.

8 Power divergences

In this section we prove that the renormalization of $\xi^{(1)}(1)$ is not affected by ultraviolet power divergences. The argument presented does not rely on any perturbative expansion and is based only on the symmetries of the lattice $SHQET$.

A given operator O can mix with lower and equal dimensional operators O' allowed by the symmetries of the (regulated) theory. The mixing coefficients are proportional to the appropriate power of the ultraviolet cut-off $1/a$ to account for the dimension. If the dimension of O and of O' are the same, the mixing coefficients contain in general logarithmic divergences, of the form $\log a$. The computation of $\xi^{(1)}(1)$ involves single and double insertions of the velocity operator $(\vec{v}\cdot\vec{D}) = Q^\dagger(\vec{v}\cdot\vec{D})Q$. The only possible linear divergence in $(\vec{v}\cdot\vec{D})$ is through the mixing with the operator $\mathbf{1} = Q^\dagger Q$

$$< (\vec{v}\cdot\vec{D}) > = \frac{k}{a} < \mathbf{1} > + \ at\ most\ logarithmic\ terms \quad (132)$$

where k is a coefficient which has a perturbative expansion in α_s. Such a mixing is however impossible because of the spatial parity invariance of the theory. Then, we have $k = 0$.

The double insertion of $(\vec{v}\cdot\vec{D})$ can also mix with $\mathbf{1}$, and in this case the mixing is not forbidden by any symmetry

$$< (\vec{v}\cdot\vec{D})\ (\vec{v}\cdot\vec{D}) > = \frac{c}{a}\ < \mathbf{1} > + \ at\ most\ logarithmic\ terms \quad (133)$$

where now $c \neq 0$ in general. In sec.5 we have checked this result with an explicit one-loop computation.

Therefore, the two- and three-point correlators defined in eqs.(14) and (16) respectively, can be written as

$$\begin{aligned} C_3^{(2)}(t,t') &= \frac{c}{a}\,(t'-t)\;C_3^{(0)}(t,t') \;+\; \textit{at most logarithmic terms} \\ C_D^{(2)}(t'-t) &= \frac{c}{a}\,(t'-t)\;C_D^{(0)}(t'-t) \;+\; \textit{at most logarithmic terms} \end{aligned} \tag{134}$$

where the superscript indicates the order in the velocity. The derivative of the Isgur-Wise function is given by the following combination of correlation functions (see eq.(19))

$$\frac{C_3^{(2)}(t,t')}{C_3^{(0)}(t,t')} - \frac{C_D^{(2)}(t'-t)}{C_D^{(0)}(t'-t)} \tag{135}$$

Substituting eqs.(134), we see that the linear divergence cancel in the expression for the $\xi^{(1)}(1)$, as anticipated. We notice that the argument given is non-perturbative, and it is confirmed by the explicit one-loop computations of the previous sections.

This argument, however, does not hold for higher derivatives of the Isgur-Wise function. In fact, consider for example, the second derivative of this function with respect to v_4. In this case, we must deal with four insertions of the operator $(\vec{v}\cdot\vec{D})$. Therefore, the correlation functions $C_{2,3}^{(4)}$ will contain a linearly divergent contribution just as $C_{2,3}^{(2)}$. From eq.(20) we see that the correlation functions $C_{2,3}^{(4)}$ enter the expression for $\xi^{(2)}(1)$ through the combination

$$C_3^{(4)}\,C_2^{(0)} \;-\; C_3^{(0)}\,C_2^{(4)} \tag{136}$$

which again is at most logarithmically divergent because the poles $1/a$ cancel out. There is, however, a second contribution to $\xi^{(2)}(1)$ that is not free from linear divergences, namely,

$$(C_2^{(2)})^2\,C_3^{(0)} \;-\; C_3^{(2)}\,C_2^{(2)}\,C_2^{(0)} \tag{137}$$

Substituting (134) into (137) we obtain that the term $1/a^2$ cancels out in this combination but that proportional to $1/a$ survives giving rise to a linear divergence as a goes to zero. Therefore, the computation of $\xi^{(2)}(1)$ requires the subtraction both from $C_2^{(2)}$ and $C_3^{(2)}$ of a linear divergence as in the case of the self-energy of a quark.

The same reasoning can be applied to higher derivatives of the Isgur-Wise function. As we increase the order of the derivative, the power of the divergence also increases and thus non-perturbative subtractions from the correlation functions of lower velocity degree are necessary to obtain reliable results from a numerical computation on the lattice.

9 Conclusions

We have studied the lattice renormalization of the effective theory for slow heavy quarks, which allows to compute the slope of the Isgur-Wise function at the normalization point, $\xi^{(1)}(1)$, with Montecarlo simulations. We showed that the lattice-continuum renormalization constant of $\xi^{(1)}(1)$ does not contain any linear ultraviolet divergence, but only a logarithmic one. This implies that the matching of $\xi^{(1)}(1)$ can be done in perturbation theory and it is not necessary to perform any non-perturbative subtraction. The lattice computation of the slope of the Isgur-Wise function using the effective theory for slow heavy quarks is therefore feasible in principle.

The one-loop lattice renormalization constants of the slow heavy quark effective theory have been computed to order v^2 together with the matching constant of $\xi^{(1)}(1)$, which relates the value of this form factor measured on the lattice to its physical counterpart in the continuum.

We have demonstrated that the effective theory for slow heavy quarks reproduces the infrared behaviour of the original (non-expanded) theory order by order in the velocity. This means that we are dealing with a consistent expansion of the effective theory for heavy quarks.

We have analysed the lattice effective theory for slow heavy quarks also to higher orders in the velocity. Unfortunately, the renormalization of the higher derivatives of the Isgur-Wise function, $\xi^{(n)}(1)$ for $n > 1$, is affected by ultraviolet power divergences. The lattice-continuum matching of $\xi^{(n)}(1)$ is therefore much more involved than in the case of $\xi^{(1)}(1)$. We stress however that the higher derivatives of the Isgur-Wise function are much less important than the first one. The renormalization problems of the slow heavy quark effective theory which arise in higher orders do not constitute therefore a serious limitation for its phenomenological applications.

We hope that the results of our analysis may encourage the scientific community to carry out the numerical simulation of $\xi^{(1)}(1)$ using the slow heavy quark effective theory. We believe that this theory can be a source of interesting physical results.

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Table captions

Table 1: Numerical values of the constants determining the continuum–lattice matching of the first derivatives of the Isgur-Wise function.

Table 2: Renormalization group (R. G.) corrections to the constants determining the continuum–lattice matching of the first derivatives of the Isgur-Wise function at the scale $\mu = \overline{m}$ for several lattice spacings a. The two-loop anomalous dimension of the heavy-heavy quark current has been properly included. A factor $\left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{4}$ multiplying all entries is understood.

Table A.1: Numerical values of three-dimensional integrals for several values of m, the order in the expansion in powers of the velocity v_3.

Table A.2: Numerical values of the constants entering the continuum–lattice matching of the heavy-quark current for several values of m, the order in the velocity expansion.

Figure captions

Figure 1: The diagrams contributing to the one-loop heavy quark self-energy with one insertion of $(\vec{v}\cdot\vec{D})$, denoted by a crossed circle. The double line represents the heavy quark with velocity $(0, 0, v_3)$.

Figure 2: The diagrams contributing to the one-loop heavy quark self-energy with two insertions of $(\vec{v}\cdot\vec{D})$, denoted by a crossed circle. The double line represents the heavy quark with velocity $(0, 0, v_3)$.

Figure 3: The non-vanishing diagrams contributing to the one-loop heavy quark self-energy with m insertions of $(\vec{v}\cdot\vec{D})$, denoted by a crossed circle. The double line represents the heavy quark with velocity $(0, 0, v_3)$.

Figure 4: The non-vanishing diagrams contributing to the one-loop heavy quark self-energy with m insertions of $(\vec{v}\cdot\vec{D})$, denoted by a crossed circle, and n insertions of the operator D_4, represented by a full circle. The double line represents the heavy quark with velocity $(0, 0, v_3)$.

Figure 5: The non-vanishing one-particle irreducible diagrams contributing to the one-loop vertex of the current $J(x)$ with one and two insertions of $(\vec{v} \cdot \vec{D})$, denoted by a crossed circle. The double line represents the heavy quark with velocity $(0, 0, v_3)$. The full line stands for a static heavy quark.

Appendix A Analytical expressions and numerical values of loop integrals

The renormalization of both the heavy-quark current and the SHQET lagrangian can be written in terms of a few three-dimensional one-loop integrals. In this appendix, we give their analytical expressions and numerical values up to $O(v_3^4)$.

We define

$$\begin{aligned}
\mathrm{Si}^{(\alpha\beta)}(m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d^3k\,\frac{\sin^2(k_3)}{B^\alpha\sqrt{[(2+B)B]^\beta}}\,\Xi(B)^m \\
\mathrm{Cs}^{(\alpha\beta)}(m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d^3k\,\frac{\cos^2(k_3/2)}{B^\alpha\sqrt{[(2+B)B]^\beta}}\,\Xi(B)^m \\
\mathrm{Id}^{(\alpha\beta)}(m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d^3k\,\frac{1}{B^\alpha\sqrt{[(2+B)B]^\beta}}\,\Xi(B)^m
\end{aligned} \tag{A.1}$$

where for the present calculation $\alpha = 0, 2$ and $\beta = 0, 1$, and

$$\Xi(B) = \frac{\sin^2(k_3)\,(2+B)}{B}\left[1+\frac{B}{\sqrt{(2+B)B}}\right]^2 = \frac{4\,\sin^2(k_3)}{(1\,-\,z_-)^2} \tag{A.2}$$

with z_- the solution of $z_-^2 \,-\, 2\,(1+B)\,z_- \,+\, 1 \,=\, 0$ with $|\, z_- \,| < 1$. Note that the function $\Xi(B)$ is infrared convergent. This fact will be used in appendix B to subtract the infrared divergent behaviour from the integrals in eq.(A.1).

Other integrals as, for example, the one with a factor $\cos(k_3)$ instead of $\cos^2(k_3/2)$ can trivially be reduced to linear combinations of the integrals defined in eq.(A.1).

Obviously, these integrals must be evaluated numerically. However, care should be taken when infrared divergences appear as in $\mathrm{Si}^{(21)}(m)$, $\mathrm{Cs}^{(11)}(m)$ and $\mathrm{Id}^{(11)}(m)$. In fact, in this case we cannot take $\lambda = 0$. The logarithmic infrared divergence must be subtracted before computing them numerically. This has been done for arbitrary m in appendix B. We refer the reader to this appendix for details. Other integrals are infrared finite for any value of m and thus can safely be computed by means of, for example, a Monte Carlo routine.

Now, we give the analytical expressions of the heavy-quark self-energy (see eq.(82))

$$\begin{aligned}
\Sigma_0^{(2m)} &= \frac{1}{4^{m-1}}\Big[\mathrm{Cs}^{(10)}(m-1) \,+\, \mathrm{Cs}^{(01)}(m-1) \\
&- \frac{1}{2}\,\mathrm{Si}^{(20)}(m-1) \,-\, \frac{1}{2}\,\mathrm{Si}^{(11)}(m-1)\Big]
\end{aligned} \tag{A.3}$$

Type	B-Factor	Velocity Power $m = 0$	$m = 1$	$m = 2$	Infr. Diver. $\log(a\lambda)$
Si	(10)	8.284	26.148	178.85	0
Si	(11)	3.367	15.20	78.8	0
Si	(01)	5.791	24.877	120.76	0
Si	(20)	6.771	29.2	148	0
Si	(21)	-0.036	9.83	78.9	$\frac{2\,4^{m+1}}{2m+3}$
Ci	(10)	13.34	34.8	149	0
Ci	(11)	2.485	12.93	72.9	$\frac{4^{m+1}}{2m+1}$
Ci	(01)	7.298	21.21	88.0	0
Id	(10)	19.95	39.2	241	0
Id	(11)	4.526	20.21	108.5	$\frac{4^{m+1}}{2m+1}$
Id	(01)	12.23	34.9	152	0

Table A.1: Numerical values of three-dimensional integrals for several values of m, the order in the expansion in powers of the velocity v_3.

Constant	Velocity Power $m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$
$\Sigma_0^{(m)}$	-19.95	0.0	15.57	0.0	8.20	0.0	7.75
$\Sigma_{40}^{(m)}$	24.48	0.0	7.60	0.0	7.55	0.0	8.65
$\Sigma_{30}^{(m)}$	0.0	12.67	0.0	7.28	0.0	8.37	0.0
$Z_Q^{(m)}$	24.48	0.0	-5.07	0.0	0.27	0.0	0.28
$\overline{Z}_Q^{(m)}$	4.53	0.0	30.45	0.0	-7.10	0.0	-0.17
$Z_v^{(m)}$	11.81	0.0	0.32	0.0	-0.82	0.0	$--$
$Z_\xi^{(m)}$	19.95	0.0	4.65	0.0	-1.59	0.0	-1.04
$\overline{Z}_\xi^{(m)}$	0.0	0.0	12.44	0.0	-5.28	0.0	2.42

Table A.2: Numerical values of the constants entering the continuum–lattice matching of the heavy-quark current for several values of m, the order in the velocity expansion.

$$
\begin{aligned}
\Sigma_{30}^{(2m+1)} &= \frac{1}{4^{m-1}} \left[\frac{2m+1}{2} \left\{ \mathrm{Si}^{(10)}(m-1) \,+\, \mathrm{Si}^{(01)}(m-1) \,+\, \mathrm{Si}^{(11)}(m-1) \right\} \theta(m) \right. \\
&- \; (2m-1) \left\{ \mathrm{Cs}^{(10)}(m-1) \,+\, \mathrm{Cs}^{(01)}(m-1) \,+\, \mathrm{Cs}^{(11)}(m-1) \right\} \theta(m) \\
&+ \left. \frac{2m+1}{4} \left\{ 2\,\mathrm{Cs}^{(11)}(m) \,-\, \mathrm{Id}^{(11)}(m) \right\} \,+\, \frac{1}{4}\,\mathrm{Id}^{(01)}(m)\,\delta_{m,0} \right] \qquad \text{(A.4)} \\
\Sigma_{40}^{(2m)} &= \frac{1}{4^{m-1}} \left[\mathrm{Si}^{(20)}(m-1) \,+\, \mathrm{Si}^{(11)}(m-1) \right. \\
&+ \left. \frac{2m+1}{2}\; \mathrm{Si}^{(21)}(m-1) \,-\, (2m-1)\,\mathrm{Cs}^{(11)}(m-1) \right] \qquad \text{(A.5)}
\end{aligned}
$$

which are supplemented with the old results for a static heavy quark

$$
\Sigma_{0}^{(0)} \,=\, -\mathrm{Id}^{(10)}(0) \qquad \Sigma_{40}^{(0)} \,=\, \mathrm{Id}^{(10)}(0) \,+\, \mathrm{Id}^{(11)}(0) \qquad \text{(A.6)}
$$

In table A.1, we list the numerical values of the three-dimensional lattice regularized integrals (A.1). These quantities has been evaluated using both a Monte Carlo and a lattice integration routine. Errors are at most $O(1)$ in the last decimal place.

In table A.2, we present the numerical values for the heavy-quark self-energy, the wave function renormalization, the velocity renormalization and the lattice-continuum matching constants for the Isgur-Wise function. As before, errors are at most $O(1)$ in the last decimal place. The constants with a bar are in the real space lattice renormalization scheme while the others has been calculated in the momentum space scheme.

Appendix B Infrared subtraction of loop integrals

Feynman integrals appearing in lattice perturbation theory must be evaluated numerically because they are too much complicate to be integrated analytically. The trouble arises when these integrals are divergent as λ, the fictitious gluon mass, goes to zero. In order to compute divergent lattice integrals, one has to subtract from the integrand an expression which has its same infrared behavior [23]. Doing so, the integral to be calculated can be expressed as a sum of an infrared finite one, in which we can safely set $\lambda = 0$, and a second integral which contains the divergences of the original one. The former can be evaluated numerically while the later must be computed analytically to explicitly display the terms which depend on the infrared and ultraviolet regulators.

In this calculation, all four-dimensional integrals can be reduced to three-dimensional ones by integrating over the zeroth component of the loop momentum or performing an integration by parts. This simplification is possible because of the simple structure of the heavy quark propagator that only depends on the zeroth component of the momentum. The integrands of the resulting three-dimensional integrals turn out to be algebraic functions of B, defined by

$$B = \sum_{\alpha=1}^{3} (1 - \cos(k_\alpha)) + \frac{\lambda^2 a^2}{2} \tag{B.1}$$

Therefore, it is convenient to know the infrared limit of B itself and some other functions of it. For $| k_\alpha | \ll 1$ we have

$$\begin{aligned}
B &\approx \frac{1}{2}(\vec{k}^2 + \lambda^2)\,[1 - \frac{1}{12}(\vec{k}^2 + \lambda^2)] \\
\frac{1}{\sqrt{(1+B)^2 - 1}} &\approx \frac{1}{\sqrt{\vec{k}^2 + \lambda^2}}\,[1 - \frac{1}{12}(\vec{k}^2 + \lambda^2)] \\
\frac{1}{B\sqrt{(1+B)^2 - 1}} &\approx \frac{2}{(\vec{k}^2 + \lambda^2)^{3/2}}\,[1 - \frac{1}{4}\lambda^2] \\
\frac{1}{B^2\sqrt{(1+B)^2 - 1}} &\approx \frac{4}{(\vec{k}^2 + \lambda^2)^{5/2}}\,[1 - \frac{1}{12}\vec{k}^2 - \frac{1}{3}\lambda^2]
\end{aligned} \tag{B.2}$$

The previous expansions are almost all we need to extract the logarithmic infrared divergence of our one-loop integrals at every order in the velocity.

As we saw in appendix A, the only infrared divergent integrals are $\mathrm{Si}^{(21)}(m)$, $\mathrm{Cs}^{(11)}(m)$ and $\mathrm{Id}^{(11)}(m)$, defined in eq.(A.1). In order to numerically compute these integrals, we find their infrared behaviour using (B.2) and then construct the corresponding regularizing integrals which are

$$\begin{aligned}
\mathrm{Si}^{(21)}(m)_{IR} &= 4^{m+1}\int_{-\pi}^{\pi} d^3k\,\frac{k_3^{2+2m}}{(\vec{k}^2 + \lambda^2)^{5/2+m}}\,\theta(\pi^2 - k^2) \\
\mathrm{Cs}^{(11)}(m)_{IR} &= 2\,4^{m}\int_{-\pi}^{\pi} d^3k\,\frac{k_3^{2m}}{(\vec{k}^2 + \lambda^2)^{3/2+m}}\,\theta(\pi^2 - k^2) \\
\mathrm{Id}^{(11)}(m)_{IR} &= 2\,4^{m}\int_{-\pi}^{\pi} d^3k\,\frac{k_3^{2m}}{(\vec{k}^2 + \lambda^2)^{3/2+m}}\,\theta(\pi^2 - k^2)
\end{aligned} \tag{B.3}$$

where we perform the integration on a 3-sphere of radius π to take advantage of spherical symmetry. Of course any other radius $R > 0$ would be equally good.

The three-dimensional integrals in (B.3) are much simpler to be evaluated than the original ones. In fact, the best thing we can do is to separate the radial and angular integrations expanding k_3 as a Gegenbauer series

$$k_3^n = k^n\,\frac{\Gamma(\nu)\,n}{2^n}\sum_{j=0}^{[n/2]}\frac{(n - 2j + \nu)}{j!\,\Gamma(1 + \nu + n - j)}\,C^{\nu}_{n-2j}(\hat{k}\cdot\hat{e}_3) \tag{B.4}$$

where ν is related to the space dimension D by $D = 2(\nu + 1)$, therefore $\nu = 1/2$ in our case. Inserting (B.4) into (B.3), we have

$$\begin{aligned}
\mathrm{Si}^{(21)}(m)_{IR} &= 4^{m+1} \frac{2}{(2m+3)} I_R(m+2, m+2) \\
\mathrm{Cs}^{(11)}(m)_{IR} &= 4^{m+1} \frac{1}{(2m+1)} I_R(m+1, m+1) \\
\mathrm{Id}^{(11)}(m)_{IR} &= 4^{m+1} \frac{1}{(2m+1)} I_R(m+1, m+1)
\end{aligned} \tag{B.5}$$

where $I_R(\alpha, \beta)$ is the following radial integral

$$I_R(\alpha, \beta) = \int_0^\pi dk \frac{k^{2\alpha}}{(\vec{k}^2 + \lambda^2)^{3/2+\beta-1}} \tag{B.6}$$

The integration limits are a consequence of the Heviside function introduced in (B.3).

The radial integral $I_R(m, m)$ can be evaluated by noting that

$$I_R(m, m) = I_R(m-1, m-1) + \frac{\lambda^2}{(3/2 + m - 2)} \frac{d}{d\lambda^2} I_R(m-1, m-1) \tag{B.7}$$

along with

$$I_R(1, 1) = \log(2\pi) - 1 - \log(a\lambda) \tag{B.8}$$

The result is

$$I_R(m, m) = \log(2\pi) - \sum_{n=0}^{m-1} \frac{1}{(2n+1)} - \log(a\lambda) \tag{B.9}$$

Putting all the formulas together, we arrive at the following infrared subtracted basic integrals

$$\begin{aligned}
\mathrm{Si}^{(21)}(m) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d^3k \left[\frac{\sin^2(k_3)}{B^2 \sqrt{(2+B)B}} \Xi^m(B) - 4^m \frac{4\, k_3^{2(m+1)}\, \theta(\pi^2 - k^2)}{\left(k^2 + \lambda^2\right)^{5/2+m}} \right] \\
&+ \frac{2\, 4^{m+1}}{(2m+3)} \left[\log(2\pi) - \sum_{n=0}^{m+1} \frac{1}{(2n+1)} - \log(a\lambda) \right]
\end{aligned} \tag{B.10}$$

$$\begin{aligned}
\mathrm{Cs}^{(11)}(m) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d^3k \left[\frac{\cos^2(k_3/2)}{B \sqrt{(2+B)B}} \Xi^m(B) - 4^m \frac{2\, k_3^{2m}\, \theta(\pi^2 - k^2)}{\left(k^2 + \lambda^2\right)^{3/2+m}} \right] \\
&+ \frac{4^{m+1}}{(2m+1)} \left[\log(2\pi) - \sum_{n=0}^{m} \frac{1}{(2n+1)} - \log(a\lambda) \right]
\end{aligned} \tag{B.11}$$

$$\begin{aligned}
\mathrm{Id}^{(11)}(m) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d^3k \left[\frac{1}{B \sqrt{(2+B)B}} \Xi^m(B) - 4^m \frac{2\, k_3^{2m}\, \theta(\pi^2 - k^2)}{\left(k^2 + \lambda^2\right)^{3/2+m}} \right] \\
&+ \frac{4^{m+1}}{(2m+1)} \left[\log(2\pi) - \sum_{n=0}^{m} \frac{1}{(2n+1)} - \log(\lambda a) \right]
\end{aligned} \tag{B.12}$$

Using the previous equations, we have calculated the numerical values of these divergent integrals which are tabulated in Table A.2. Moreover, the coefficients of the logarithmic divergence determine the anomalous dimension of the heavy-heavy quark current and the running of the derivatives of the Isgur-Wise function.

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